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# (Hierarchical) Identity-Based Encryption from Affine Message Authentication

Olivier Blazy

Eike Kiltz

Jiaxin Pan

Faculty of Mathematics  
Horst Görtz Institute for IT-Security  
Ruhr University Bochum, Germany  
`{olivier.blazy,eike.kiltz,jiaxin.pan}@rub.de`

## Abstract

We provide a generic transformation from any *affine* message authentication code (MAC) to an identity-based encryption (IBE) scheme over pairing groups of prime order. If the MAC satisfies a security notion related to unforgeability against chosen-message attacks and, for example, the  $k$ -Linear assumption holds, then the resulting IBE scheme is adaptively secure. Our security reduction is tightness preserving, i.e., if the MAC has a tight security reduction so has the IBE scheme. Furthermore, the transformation also extends to hierarchical identity-based encryption (HIBE). We also show how to construct affine MACs with a tight security reduction to standard assumptions. This, among other things, provides the first tightly secure HIBE in the standard model.

**Keywords:** IBE, HIBE, standard model, tight reduction

## 1 Introduction

Identity-based encryption (IBE) [26] enables a user to encrypt to a recipient's identity  $\text{id}$  (e.g., an email or phone number), and decryption can be done using a user secret key for  $\text{id}$ , obtained by a trusted authority. The first instantiations of an IBE scheme were given in 2001 [8, 5, 25]. Whereas earlier constructions relied on the random oracle model, the first adaptively secure construction in the standard model was proposed in [28]. Here adaptive security means that an adversary may select the challenge identity  $\text{id}^*$  after seeing the public key and arbitrarily many user secret keys for identities of his choice. The concept of IBE generalizes naturally to hierarchical IBE (HIBE). In an  $L$ -level HIBE, hierarchical identities are vectors of identities of maximal length  $L$  and user secret keys for a hierarchical identity can be delegated. An IBE is simply a  $L$ -level HIBE with  $L = 1$ .

In this work we focus on adaptively secure (H)IBE schemes in the standard model. The construction from [28] has the disadvantage of a non-tight security reduction, i.e., the security reduction reducing security of the  $L$ -level HIBE to the hardness of the underlying assumption loses at least a factor of  $Q^L$ , where  $Q$  is the maximal number of user secret key queries. Modern HIBE schemes [27, 7] only lose a factor  $Q$ , independent of  $L$ . The first tightly secure IBE was recently proposed by Chen and Wee [7] but designing a  $L$ -level HIBE for  $L > 1$  and a tight (i.e., independent of  $Q$ ) security reduction to a standard assumption remains an open problem.

Until now, all known constructions of (H)IBE schemes are specific, i.e., they are custom-made to a specific hardness assumption. This is in contrast to other basic cryptographic primitives such as signatures and public-key encryption, for which efficient generic transformations have been known for a long time. We would like to highlight the concept of smooth projective hash proof systems for chosen-ciphertext secure encryption [10] and an old construction by Bellare and Goldwasser [2] that transforms any pseudorandom function (PRF) plus a non-interactive zero-knowledge (NIZK) proof into a signature

scheme. Until today no generic construction of a (H)IBE from any “simple” low-level cryptographic primitive is known. However, the recent IBE scheme by Chen and Wee [7] uses a specific randomized PRF at the core of their construction, but its usage is non-modular.

## 1.1 This work

**AFFINE MACs.** In this work we put forward the notion of *affine message authentication codes* (affine MACs). An affine MAC over  $\mathbb{Z}_q^n$  is a randomized MAC with a special algebraic structure over some group  $\mathbb{G} = \langle g \rangle$  of prime-order  $q$ . For a vector  $\mathbf{a} \in \mathbb{Z}_q^n$ , define  $[\mathbf{a}] := g^{\mathbf{a}} = (g^{a_1}, \dots, g^{a_n})^\top \in \mathbb{G}^n$  as the implicit representation of  $\mathbf{a}$  over  $\mathbb{G}$ . Roughly speaking, the MAC tag  $\tau_{\mathbf{m}} = ([\mathbf{t}], [u])$  of an affine MAC over  $\mathbb{Z}_q^n$  on message  $\mathbf{m} \in \mathcal{M}$  is split into a random message-independent part  $[\mathbf{t}] \in \mathbb{G}^n$  plus a message-depending affine part  $[u] \in \mathbb{G}$  satisfying

$$u = \sum f_i(\mathbf{m}) \mathbf{x}_i^\top \cdot \mathbf{t} + \sum f'_i(\mathbf{m}) x'_i \in \mathbb{Z}_q, \quad (1)$$

where  $f_i, f'_i : \mathcal{M} \rightarrow \mathbb{Z}_q$  are public functions and  $\mathbf{x}_i \in \mathbb{Z}_q^n, x'_i \in \mathbb{Z}_q$  are from the secret key  $\mathbf{sk}_{\text{MAC}}$ . Almost all group-based MACs recently considered in [11], as well as the MAC derived from the randomized Naor-Reingold PRF [23] implicitly given in [7] are affine.

**FROM AFFINE MACs TO IBE.** Let us fix (possibly symmetric) pairing groups  $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$  equipped with a bilinear map  $e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_T$ . Let  $\mathcal{D}_k$ -MDDH be any Matrix Diffie-Hellman Assumption [12]\* that holds in  $\mathbb{G}_1$ , e.g.,  $k$ -Linear or DDH.

Our main result is a *generic transformation*  $\text{IBE}[\text{MAC}_n, \mathcal{D}_k]$  from any affine message authentication code  $\text{MAC}_n$  over  $\mathbb{Z}_q^n$  into an IBE scheme. If  $\text{MAC}_n$  (defined over  $\mathbb{G}_2$ ) is PR-CMA-secure (pseudorandom against chosen message attacks, a decisional variant of the standard UF-CMA security for MACs) and the  $\mathcal{D}_k$ -MDDH assumption holds in  $\mathbb{G}_1$ , then  $\text{IBE}[\text{MAC}_n, \mathcal{D}_k]$  is an adaptively secure (and anonymous) IBE scheme. Furthermore, the security reduction of  $\text{IBE}[\text{MAC}_n, \mathcal{D}_k]$  is as tight as the one of  $\text{MAC}_n$ . The size of the public IBE parameters depends on the size of the MAC secret key  $\mathbf{sk}_{\text{MAC}}$ , whereas the IBE ciphertexts and user secret keys always contain  $n + k + 1$  group elements. We stress that our transformation works with any  $k \geq 1$  and any  $\mathcal{D}_k$ -MDDH Assumption, hence  $\mathcal{D}_k$  can be chosen to match the security assumption of  $\text{MAC}_n$ .

We also extend our generic transformation to HIBE schemes. In particular, we have two generic HIBE constructions depending on different properties of the underlying affine MACs. If the affine MAC is *delegatable* (to be defined in Section 5.1), we obtain an adaptively secure  $L$ -level HIBE  $\text{HIBE}[\text{MAC}_n, \mathcal{D}_k]$ . Furthermore, if the affine MAC is delegatable and *anonymity-preserving* (to be defined in Section 5.5), we obtain an *anonymous* and adaptively secure  $L$ -level HIBE  $\text{AHIBE}[\text{MAC}_n, \mathcal{D}_k]$ . Both of the constructions have the same tightness properties as the MAC, and their ciphertexts sizes are the same as in the IBE case. Due to different delegation methods,  $\text{AHIBE}[\text{MAC}_n, \mathcal{D}_k]$  has slightly shorter public parameters, but larger user secret keys than  $\text{HIBE}[\text{MAC}_n, \mathcal{D}_k]$ .

Let us highlight again the fact that the underlying object is a *symmetric* primitive (a MAC) that we transform to an *asymmetric* primitive (an IBE scheme). Furthermore, as a MAC is a very simple and well-understood object, we hope that our transformation can contribute to understanding the more complex object of an IBE scheme.

**TWO DELEGATABLE AFFINE MACs.** To instantiate our transformations, we consider two specific delegatable affine MACs. Our first construction,  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$ , is a generalization of the MAC derived from the randomized Naor-Reingold PRF [7] to any  $\mathcal{D}_k$ -MDDH Assumption. (Unfortunately, the MAC based on the original deterministic Naor-Reingold PRF is not affine.) We show that it is affine over  $\mathbb{Z}_q^n$  with  $n = k$  and delegatable. We prove PR-CMA-security with an (almost) tight security reduction to  $\mathcal{D}_k$ -MDDH. (Almost tight, as the security reduction loses a factor  $O(m)$ , where  $m$  is the length of the message space.) This leads to the first HIBE with a tight security reduction to a standard assumption. Ciphertexts and user secret keys of  $\text{HIBE}[\text{MAC}_{\text{NR}}[\mathcal{D}_k], \mathcal{D}_k]$  only contain  $2k + 1$  group elements which is 3 in case we use

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\*The  $\mathcal{D}_k$ -MDDH assumption over  $\mathbb{G}_1$  captures naturally all subspace decisional assumptions over prime order groups. Concretely, it states that given  $[\mathbf{A}]_1 \in \mathbb{G}_1^{(k+1) \times k}$ , the value  $[\mathbf{A} \cdot \mathbf{w}]_1 \in \mathbb{G}_1^{k+1}$  is pseudorandom, where  $\mathbf{A} \in \mathbb{Z}_q^{(k+1) \times k}$  gets chosen according to distribution  $\mathcal{D}_k$  and  $\mathbf{w} \in \mathbb{Z}_q^k$ . Examples include  $k$ -Linear and DDH ( $k = 1$ ).

Scheme	$ \mathbf{pk} $	$ \mathbf{usk} $	$ \mathbf{C} $	Anon.	Loss	Assumption
Wat05 [28]	$(4 + \lambda) \mathbb{G}_1 $	$2 \mathbb{G}_2 $	$2 \mathbb{G}_1 $	–	$O(\lambda Q)$	DBDH
Wat09 [27]	$12 \mathbb{G}_1  +  \mathbb{G}_T $	$8 \mathbb{G}_2  +  \mathbb{Z}_q $	$9 \mathbb{G}_1  +  \mathbb{Z}_q $	–	$O(Q)$	2-LIN
Lew12 [19]	$24 \mathbb{G}_1  +  \mathbb{G}_T $	$6 \mathbb{G}_2 $	$6 \mathbb{G}_1 $	✓	$O(Q)$	2-LIN
CLL <sup>+</sup> 12 [6]	$8 \mathbb{G}_1  +  \mathbb{G}_T $	$4 \mathbb{G}_2 $	$4 \mathbb{G}_1 $	–	$O(Q)$	SXDH
JR13 [16]	$6 \mathbb{G}_1  +  \mathbb{G}_T $	$5 \mathbb{G}_2 $	$3 \mathbb{G}_1  +  \mathbb{Z}_q $	–	$O(Q)$	SXDH
CW13 [7]	$2k^2(2\lambda + 1) \mathbb{G}_1  + k \mathbb{G}_T $	$4k \mathbb{G}_2 $	$4k \mathbb{G}_1 $	–	$O(\lambda)$	$k$ -LIN
IBE <sub>HPS</sub>	$(3k^2 + 4k) \mathbb{G}_1 $	$(2k + 2) \mathbb{G}_2 $	$(2k + 2) \mathbb{G}_1 $	✓	$O(Q)$	$k$ -LIN
IBE <sub>NR</sub>	$(2\lambda k^2 + 2k) \mathbb{G}_1 $	$(2k + 1) \mathbb{G}_2 $	$(2k + 1) \mathbb{G}_1 $	✓	$O(\lambda)$	$k$ -LIN
Wat05 [28]	$O(\lambda L) \mathbb{G}_1 $	$O(\lambda L) \mathbb{G}_2 $	$(1 + L) \mathbb{G}_1 $	–	$O(\lambda Q)^L$	DBDH
Wat09 [27]	$O(L) \mathbb{G}_1 $	$O(L)( \mathbb{G}_2  +  \mathbb{Z}_q )$	$O(L)( \mathbb{G}_1  +  \mathbb{Z}_q )$	–	$O(Q)$	2-LIN
CW13 [7]	$O(Lk^2)( \mathbb{G}_1  +  \mathbb{G}_2 )$	$O(Lk) \mathbb{G}_2 $	$(2k + 2) \mathbb{G}_1 $	–	$O(Q)$	$k$ -LIN
HIBE <sub>HPS</sub>	$O(Lk^2)( \mathbb{G}_1  +  \mathbb{G}_2 )$	$O(Lk) \mathbb{G}_2 $	$(2k + 2) \mathbb{G}_1 $	–	$O(Q)$	$k$ -LIN
HIBE <sub>NR</sub>	$O(L\lambda k^2)( \mathbb{G}_1  +  \mathbb{G}_2 )$	$O(L\lambda k) \mathbb{G}_2 $	$(2k + 1) \mathbb{G}_1 $	–	$O(L\lambda)$	$k$ -LIN
AHIBE <sub>HPS</sub>	$O(Lk^2) \mathbb{G}_1 $	$O(Lk^2) \mathbb{G}_2 $	$(2k + 2) \mathbb{G}_1 $	✓	$O(Q)$	$k$ -LIN

**Table 1:** Top: comparison between known adaptively secure IBEs with identity-space  $\mathcal{ID} = \{0, 1\}^\lambda$  in prime order groups from standard assumptions. We count the number of group elements in  $\mathbb{G}_1, \mathbb{G}_2$ , and  $\mathbb{G}_T$ .  $Q$  is the number of user secret key queries by the adversary. ‘Anon.’ stands for anonymity. Here  $\text{IBE}_{\text{HPS}} := \text{IBE}[\text{MAC}_{\text{HPS}}[\mathcal{D}_k], k\text{-LIN}]$ ,  $\text{IBE}_{\text{NR}} := \text{IBE}[\text{MAC}_{\text{NR}}[\mathcal{D}_k], k\text{-LIN}]$  are from this paper. Bottom: comparison of  $L$ -level HIBEs with identity-space  $\mathcal{ID} = (\{0, 1\}^\lambda)^L$ .  $\text{HIBE}_{\text{HPS}} := \text{HIBE}[\text{MAC}_{\text{HPS}}[\mathcal{D}_k], k\text{-LIN}]$ ,  $\text{HIBE}_{\text{NR}} := \text{HIBE}[\text{MAC}_{\text{NR}}[\mathcal{D}_k], k\text{-LIN}]$  and  $\text{AHIBE}_{\text{HPS}} := \text{AHIBE}[\text{MAC}_{\text{HPS}}[\mathcal{D}_k], k\text{-LIN}]$  are from this paper.

$k = 1$  and the SXDH Assumption (i.e., DDH in  $\mathbb{G}_1$  and  $\mathbb{G}_2$ ). Interestingly, our SXDH-based IBE scheme (given explicitly in Appendix D) can be seen as a “two-copy version” of Waters’ IBE [28] which does not have a tight security reduction. The disadvantage of  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$  is that the public parameters of  $\text{IBE}[\text{MAC}_{\text{NR}}[\mathcal{D}_k], \mathcal{D}_k]$  are linear in the bit-size of the identity space.

Our second construction,  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$ , is based on a hash proof system given in [12] for any  $\mathcal{D}_k$ -MDDH problem. A hash proof system is known to imply a UF-CMA-secure MAC [11]. We extend this result to PR-CMA-security, where the reduction loses a factor of  $Q$ , the number of MAC queries. Furthermore,  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  is affine over  $\mathbb{Z}_q^{k+1}$  (i.e.,  $n = k + 1$ ) and delegatable. Whereas public parameters of the  $L$ -level HIBE  $\text{HIBE}[\text{MAC}_{\text{HPS}}[\mathcal{D}_k], \mathcal{D}_k]$  only depend on  $L$ , ciphertexts and user secret keys contain  $2k + 2$  group elements which is 4 in case of the SXDH assumption ( $k = 1$ ). We remark that the efficiency of  $\text{HIBE}[\text{MAC}_{\text{HPS}}[\mathcal{D}_k], \mathcal{D}_k]$  is roughly the same as a HIBE proposed in [7]. Additionally, we show  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  is also anonymity-preserving, which implies an anonymous (but non-tight) HIBE,  $\text{AHIBE}[\text{MAC}_{\text{HPS}}[\mathcal{D}_k], \mathcal{D}_k]$ , while the delegatable  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$  is unlikely to be anonymity-preserving.

Table 1 summarizes all known (H)IBE scheme and their parameters.

EXTENSIONS. In fact, our generic transformation even gives (hierarchical) ID-based hash proof system from any (delegatable) affine MAC and the  $\mathcal{D}_k$ -MDDH assumption. From an (H)ID-based hash proof system one readily obtains an IND-ID-CCA-secure (H)IBE [18]. Furthermore, any (H)IBE directly implies a (Hierarchical ID-based) signature scheme [13, 17]. The signature obtained from  $\text{IBE}[\text{MAC}_{\text{NR}}[\mathcal{D}_k], \mathcal{D}_k]$  has a tight security reduction. Even though it is not entirely structure preserving, it can still be used to obtain a constant-size IND-CCA-secure public-key encryption scheme with a tight security reduction in the multi-user and multi-challenge setting [15, 4].

## 1.2 Technical details

OUR TRANSFORMATION. The high level idea behind our generic transformation  $\text{IBE}[\text{MAC}, \mathcal{D}_k]$  from any affine MAC over  $\mathbb{Z}_q^n$  to an IBE scheme is the transformation from Bellare and Goldwasser [2] from a MAC (originally, a PRF) and a NIZK to a signature scheme. We use the same approach but define the user secret keys to be Bellare-Goldwasser signatures. The (H)IBE encryption functionality makes use of the special properties of the algebraic MAC and (tuned) Groth-Sahai proofs.

Concretely, the public key  $\mathbf{pk}$  of the IBE scheme contains special perfectly hiding commitments  $[\mathbf{Z}]_1$  to the MAC secret keys  $\mathbf{sk}_{\text{MAC}}$ , which also depend on the  $\mathcal{D}_k$ -MDDH assumption. The user secret key

$\text{usk}[\text{id}]$  of an identity  $\text{id}$  contains the MAC tag  $\tau_{\text{id}} = ([t]_2, [u]_2) \in \mathbb{G}_2^{n+1}$  on  $\text{id}$ , plus a tuned Groth-Sahai [14] non-interactive zero-knowledge (NIZK) proof  $\pi$  that  $\tau_{\text{id}}$  was computed correctly with respect to the commitments  $[\mathbf{Z}]_1$  containing  $\text{sk}_{\text{MAC}}$ . Since the MAC is affine, the NIZK proof  $\pi \in \mathbb{G}^k$  is very compact. The next observation is that the NIZK verification equation for  $\pi$  is a linear equation in the (committed) MAC secret keys and hence a randomized version of it gives rise to the IBE ciphertext and a decryption algorithm.

**SECURITY PROOF.** The security proof can also be sketched easily at a high level. We first apply a Cramer-Shoup argument [9], where we decrypt the IBE challenge ciphertext using the MAC secret key  $\text{sk}_{\text{MAC}}$ . Next, we make the challenge ciphertext inconsistent which involves one application of the  $\mathcal{D}_k$ -MDDH assumption. Now we can use the NIZK simulation routine to simulate the NIZK proof  $\pi$  from the user secret key  $\text{usk}[\text{id}] = (\tau_{\text{id}}, \pi)$ . At this point, as the commitments perfectly hide the MAC secret keys  $\text{sk}_{\text{MAC}}$ , the only part of the security experiment still depending on  $\text{sk}_{\text{MAC}}$  is  $\tau_{\text{id}}$  from  $\text{usk}[\text{id}]$  plus the computation of the challenge ciphertext. Now we are in the position to make the reduction to the symmetric primitive. We can use the PR-CMA symmetric security of MAC to argue directly about the pseudorandomness of the IBE challenge ciphertext. An IBE with pseudorandom ciphertexts is both IND-CPA secure and anonymous.

### 1.3 Other related work

Recently, Wee [29] proposed an information-theoretic primitive called *predicate encodings* that characterize the underlying algebraic structure of a number of predicate encoding schemes, including known IBE [21] and attribute-based encryption (ABE) [20] schemes. The main conceptual difference to affine MACs is that predicate encodings is a purely information-theoretic object. Furthermore, the framework by Wee is inherently limited to composite order groups.

Waters introduced the dual system framework [27] in order to facility tighter proofs for (H)IBE systems and beyond. The basic idea is that there exists functional and semi-functional ciphertexts and user secret keys, that are computationally indistinguishable. Decrypting a ciphertext with a user secret key is successful unless both are semi-functional. The  $\mathcal{D}_k$ -MDDH assumptions are specifically tailored to the dual system framework as they provide natural subspace assumptions over  $\mathbb{G}^{k+1}$ . Previous dual system constructions [27, 21, 7] usually first construct a scheme over composite-order groups and then transform it into prime-order groups. As the transformation uses a subspace assumption over  $\mathbb{G}^{k+1}$  for *each component* of the composite-order group, ciphertexts and user secret keys contain at least  $2(k+1)$  group elements. An exception is a recent *direct construction* in prime-order groups by Jutla and Roy [16]. Their scheme is based on the SXDH assumption (*i.e.*,  $k=1$ ) and achieves slightly better ciphertext size of 3 group elements plus one element from  $\mathbb{Z}_q$ . Even though our construction and proof strategy is inspired by the Bellare-Goldwasser NIZK approach and Cramer-Shoup’s hash proof systems, we still roughly follow the dual system framework. However, as we give a direct construction in prime-order groups, our IBE scheme  $\text{IBE}[\text{MAC}_{\text{NR}}[\mathcal{D}_k], \mathcal{D}_k]$  has ciphertexts and user secret keys of size  $2k+1$ , breaking the “ $2(k+1)$  barrier”.

Lewko and Waters [22] consider the difficulty of a security proof for  $L$ -level HIBEs that does not proving exponentially in  $L$ . Essentially, they prove that any scheme with rerandomizable user secret keys (over the space of all “functional” user secret keys) will suffer an exponential degradation in security. While some of our tightly-secure HIBEs are rerandomizable, they are only rerandomizable over the space of all user secret keys generated by the user secret key generation algorithm. Hence, our tightly-secure HIBE does not contradict the negative results of [22].

### 1.4 Open problems

We leave finding a PR-CMA-secure algebraic MAC with a tight security reduction and constant-size secret keys as an open problem. Given our main result this would directly imply a tightly-secure (H)IBE with constant-size public parameters. Furthermore, we leave finding a tightly-secure and anonymity-preserving delegatable affine MAC as an open problem, which would imply a tightly-secure anonymous HIBE.

Finally, we think that the concept of algebraic MACs can be extended such that our transformation also covers more general predicate encoding schemes, including attribute-based encryption.

## 2 Definitions

### 2.1 Notation

If  $\mathbf{x} \in \mathcal{B}^n$ , then  $|\mathbf{x}|$  denotes the length  $n$  of the vector. Further,  $x \leftarrow_s \mathcal{B}$  denotes the process of sampling an element  $x$  from set  $\mathcal{B}$  uniformly at random. If  $\mathbf{A} \in \mathbb{Z}_q^{(k+1) \times n}$  is a matrix, then  $\bar{\mathbf{A}} \in \mathbb{Z}_q^{k \times n}$  denotes the upper matrix of  $\mathbf{A}$  and then  $\underline{\mathbf{A}} \in \mathbb{Z}_q^{1 \times k}$  denotes the last row of  $\mathbf{A}$ .

**GAMES.** We use games for our security reductions. A game  $\mathcal{G}$  is defined by procedures INITIALIZE and FINALIZE, plus some optional procedures  $P_1, \dots, P_n$ . All procedures are given using pseudo-code, where initially all variables are undefined. An adversary  $\mathcal{A}$  is executed in game  $\mathcal{G}$  if it first calls INITIALIZE, obtaining its output. Next, it may make arbitrary queries to  $P_i$  (according to their specification), again obtaining their output. Finally, it makes one single call to FINALIZE( $\cdot$ ) and stops. We define  $\mathcal{G}^{\mathcal{A}}$  as the output of  $\mathcal{A}$ 's call to FINALIZE.

### 2.2 Pairing groups and Matrix Diffie-Hellman Assumption

Let  $\mathcal{G}\text{Gen}$  be a probabilistic polynomial time (PPT) algorithm that on input  $1^\lambda$  returns a description  $\mathcal{G} = (\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, q, g_1, g_2, e)$  of asymmetric pairing groups where  $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$  are cyclic groups of order  $q$  for a  $\lambda$ -bit prime  $q$ ,  $g_1$  and  $g_2$  are generators of  $\mathbb{G}_1$  and  $\mathbb{G}_2$ , respectively, and  $e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_T$  is an efficiently computable (non-degenerated) bilinear map. Define  $g_T := e(g_1, g_2)$ , which is a generator in  $\mathbb{G}_T$ .

We use implicit representation of group elements as introduced in [12]. For  $s \in \{1, 2, T\}$  and  $a \in \mathbb{Z}_q$  define  $[a]_s = g_s^a \in \mathbb{G}_s$  as the *implicit representation* of  $a$  in  $\mathbb{G}_s$ . More generally, for a matrix  $\mathbf{A} = (a_{ij}) \in \mathbb{Z}_q^{n \times m}$  we define  $[\mathbf{A}]_s$  as the implicit representation of  $\mathbf{A}$  in  $\mathbb{G}_s$ :

$$[\mathbf{A}]_s := \begin{pmatrix} g_s^{a_{11}} & \dots & g_s^{a_{1m}} \\ g_s^{a_{n1}} & \dots & g_s^{a_{nm}} \end{pmatrix} \in \mathbb{G}_s^{n \times m}$$

We will always use this implicit notation of elements in  $\mathbb{G}_s$ , i.e., we let  $[a]_s \in \mathbb{G}_s$  be an element in  $\mathbb{G}_s$ . Note that from  $[a]_s \in \mathbb{G}_s$  it is generally hard to compute the value  $a$  (discrete logarithm problem in  $\mathbb{G}_s$ ). Further, from  $[b]_T \in \mathbb{G}_T$  it is hard to compute the value  $[b]_1 \in \mathbb{G}_1$  and  $[b]_2 \in \mathbb{G}_2$  (pairing inversion problem). Obviously, given  $[a]_s \in \mathbb{G}_s$  and a scalar  $x \in \mathbb{Z}_q$ , one can efficiently compute  $[ax]_s \in \mathbb{G}_s$ . Further, given  $[a]_1, [a]_2$  one can efficiently compute  $[ab]_T$  using the pairing  $e$ . For  $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_q^k$  define  $e([\mathbf{a}]_1, [\mathbf{b}]_2) := [\mathbf{a}^\top \mathbf{b}]_T \in \mathbb{G}_T$ .

We recall the definition of the matrix Diffie-Hellman (MDDH) assumption [12].

**Definition 2.1 (Matrix Distribution)** Let  $k \in \mathbb{N}$ . We call  $\mathcal{D}_k$  a matrix distribution if it outputs matrices in  $\mathbb{Z}_q^{(k+1) \times k}$  of full rank  $k$  in polynomial time.

Without loss of generality, we assume the first  $k$  rows of  $\mathbf{A} \leftarrow_s \mathcal{D}_k$  form an invertible matrix. The  $\mathcal{D}_k$ -Matrix Diffie-Hellman problem is to distinguish the two distributions  $([\mathbf{A}], [\mathbf{A}\mathbf{w}])$  and  $([\mathbf{A}], [\mathbf{u}])$  where  $\mathbf{A} \leftarrow_s \mathcal{D}_k$ ,  $\mathbf{w} \leftarrow_s \mathbb{Z}_q^k$  and  $\mathbf{u} \leftarrow_s \mathbb{Z}_q^{k+1}$ .

**Definition 2.2 ( $\mathcal{D}_k$ -Matrix Diffie-Hellman Assumption  $\mathcal{D}_k$ -MDDH)** Let  $\mathcal{D}_k$  be a matrix distribution and  $s \in \{1, 2, T\}$ . We say that the  $\mathcal{D}_k$ -Matrix Diffie-Hellman ( $\mathcal{D}_k$ -MDDH) Assumption holds relative to  $\mathcal{G}\text{Gen}$  in group  $\mathbb{G}_s$  if for all PPT adversaries  $\mathcal{D}$ ,

$$\text{Adv}_{\mathcal{D}_k, \mathcal{G}\text{Gen}}(\mathcal{D}) := |\Pr[\mathcal{D}(\mathcal{G}, [\mathbf{A}]_s, [\mathbf{A}\mathbf{w}]_s) = 1] - \Pr[\mathcal{D}(\mathcal{G}, [\mathbf{A}]_s, [\mathbf{u}]_s) = 1]| = \text{negl}(\lambda),$$

where the probability is taken over  $\mathcal{G} \leftarrow_s \mathcal{G}\text{Gen}(1^\lambda)$ ,  $\mathbf{A} \leftarrow_s \mathcal{D}_k$ ,  $\mathbf{w} \leftarrow_s \mathbb{Z}_q^k$ ,  $\mathbf{u} \leftarrow_s \mathbb{Z}_q^{k+1}$ .

For each  $k \geq 1$ , [12] specifies distributions  $\mathcal{L}_k, \mathcal{C}_k, \mathcal{SC}_k, \mathcal{IL}_k$  such that the corresponding  $\mathcal{D}_k$ -MDDH assumption is the  $k$ -Linear assumption, the  $k$ -Cascade, the  $k$ -Symmetric Cascade, and the Incremental  $k$ -Linear Assumption, respectively. All assumptions are generically secure in bilinear groups and form

a hierarchy of increasingly weaker assumptions. The distributions are exemplified for  $k = 2$ , where  $a_1, \dots, a_6 \leftarrow_{\mathbb{S}} \mathbb{Z}_q$ .

$$\mathcal{C}_2 : \mathbf{A} = \begin{pmatrix} a_1 & 0 \\ 1 & a_2 \\ 0 & 1 \end{pmatrix} \quad \mathcal{SC}_2 : \mathbf{A} = \begin{pmatrix} a_1 & 0 \\ 1 & a_1 \\ 0 & 1 \end{pmatrix} \quad \mathcal{L}_2 : \mathbf{A} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ 1 & 1 \end{pmatrix} \quad \mathcal{U}_2 : \mathbf{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{pmatrix}.$$

It was also shown in [12] that  $\mathcal{U}_k$ -MDDH is implied by all other  $\mathcal{D}_k$ -MDDH assumptions. If  $\mathbf{A}$  is chosen from  $\mathcal{SC}_k$ , then  $[\mathbf{A}]_s$  can be represented with 1 group element; if  $\mathbf{A}$  is chosen from  $\mathcal{L}_k$  or  $\mathcal{C}_k$ , then  $[\mathbf{A}]_s$  can be represented with  $k$  group elements; If  $\mathbf{A}$  is chosen from  $\mathcal{U}_k$ , then  $[\mathbf{A}]_s$  can be represented with  $(k+1)k$  group elements. Hence,  $\mathcal{SC}_k$ -MDDH offers the same security guarantees as  $k$ -Linear, while having the advantage of a more compact representation.

Let  $m \geq 1$ . For  $\mathbf{W} \leftarrow_{\mathbb{S}} \mathbb{Z}_q^{k \times m}$ ,  $\mathbf{U} \leftarrow_{\mathbb{S}} \mathbb{Z}_q^{(k+1) \times m}$ , consider the  $m$ -fold  $\mathcal{D}_k$ -MDDH problem which is distinguishing the distributions  $([\mathbf{A}], [\mathbf{AW}])$  and  $([\mathbf{A}], [\mathbf{U}])$ . That is, the  $m$ -fold  $\mathcal{D}_k$ -MDDH problem contains  $m$  independent instances of the  $\mathcal{D}_k$ -MDDH problem (with the same  $\mathbf{A}$  but different  $\mathbf{w}_i$ ). By a hybrid argument one can show that the two problems are equivalent, where the reduction loses a factor  $m$ . The following lemma gives a tight reduction.

**Lemma 2.3 (Random self reducibility [12])** *For any matrix distribution  $\mathcal{D}_k$ ,  $\mathcal{D}_k$ -MDDH is random self-reducible. In particular, for any  $m \geq 1$ ,*

$$\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{D}) + \frac{1}{q-1} \geq \text{Adv}_{\mathcal{D}_k, \text{GGen}}^m(\mathcal{D}') := \Pr[\mathcal{D}'(\mathcal{G}, [\mathbf{A}], [\mathbf{AW}]) \Rightarrow 1] - \Pr[\mathcal{D}'(\mathcal{G}, [\mathbf{A}], [\mathbf{U}]) \Rightarrow 1],$$

with  $\mathcal{G} \leftarrow \text{GGen}(1^\lambda)$ ,  $\mathbf{A} \leftarrow_{\mathbb{S}} \mathcal{D}_k$ ,  $\mathbf{W} \leftarrow_{\mathbb{S}} \mathbb{Z}_q^{k \times m}$ ,  $\mathbf{U} \leftarrow_{\mathbb{S}} \mathbb{Z}_q^{(k+1) \times m}$ .

### 3 Message Authentication Codes

We use the standard definition of a (randomized) message authentication code  $\text{MAC} = (\text{Gen}_{\text{MAC}}, \text{Tag}, \text{Ver})$ , where  $\text{sk}_{\text{MAC}} \leftarrow_{\mathbb{S}} \text{Gen}_{\text{MAC}}(\text{par})$  returns a secret key,  $\tau \leftarrow_{\mathbb{S}} \text{Tag}(\text{sk}_{\text{MAC}}, \mathbf{m})$  returns a tag  $\tau$  on message  $\mathbf{m}$  from some message space  $\mathcal{M}$ , and  $\text{Ver}(\text{sk}_{\text{MAC}}, \mathbf{m}, \tau) \in \{0, 1\}$  returns a verification bit.

#### 3.1 Affine MACs

Affine MACs over  $\mathbb{Z}_q^n$  are group-based MACs with a specific algebraic structure.

**Definition 3.1** *Let  $\text{par}$  be system parameters containing a group  $\mathcal{G} = (\mathbb{G}_2, q, g_2)$  of prime-order  $q$  and let  $n \in \mathbb{N}$ . We say that  $\text{MAC} = (\text{Gen}_{\text{MAC}}, \text{Tag}, \text{Ver})$  is affine over  $\mathbb{Z}_q^n$  if the following conditions hold:*

1.  $\text{Gen}_{\text{MAC}}(\text{par})$  returns  $\text{sk}_{\text{MAC}}$  containing  $(\mathbf{B}, \mathbf{x}_0, \dots, \mathbf{x}_\ell, x'_0, \dots, x'_{\ell'})$ , where  $\mathbf{B} \in \mathbb{Z}_q^{n \times n'}$ ,  $\mathbf{x}_i \in \mathbb{Z}_q^n$ ,  $x'_j \in \mathbb{Z}_q$ , for some  $n', \ell, \ell' \in \mathbb{N}$ . We assume  $\mathbf{B}$  has rank at least one.
2.  $\text{Tag}(\text{sk}_{\text{MAC}}, \mathbf{m} \in \mathcal{B}^\ell)$  returns a tag  $\tau = ([\mathbf{t}]_2, [u]_2) \in \mathbb{G}_2^n \times \mathbb{G}_2$ , computed as

$$\mathbf{t} = \mathbf{B}\mathbf{s} \in \mathbb{Z}_q^n \quad \text{for } \mathbf{s} \leftarrow_{\mathbb{S}} \mathbb{Z}_q^{n'} \quad (2)$$

$$u = \sum_{i=0}^{\ell} f_i(\mathbf{m}) \mathbf{x}_i^\top \mathbf{t} + \sum_{i=0}^{\ell'} f'_i(\mathbf{m}) x'_i \in \mathbb{Z}_q \quad (3)$$

for some public defining functions  $f_i : \mathcal{M} \rightarrow \mathbb{Z}_q$  and  $f'_i : \mathcal{M} \rightarrow \mathbb{Z}_q$ . Vector  $\mathbf{t}$  is the randomness and  $u$  is the (deterministic) message-dependent part.

3.  $\text{Ver}(\text{sk}_{\text{MAC}}, \mathbf{m}, \tau = ([\mathbf{t}]_2, [u]_2))$  verifies if (3) holds.

The standard security notion for probabilistic MACs is unforgeability against chosen-message attacks UF-CMA [11]. In this work we require *pseudorandom against chosen-message attacks* (PR-CMA), which is slightly stronger than UF-CMA. Essentially, we require that the values used for one single verification equation (3) on message  $\mathbf{m}^*$  are pseudorandom over  $\mathbb{G}_1$  and  $\mathbb{G}_T$ .

<u>INITIALIZE:</u> $\text{sk}_{\text{MAC}} \leftarrow_{\$} \text{Gen}_{\text{MAC}}(\text{par})$ Return $\varepsilon$  <u>EVAL(m):</u> $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{m\}$ Return $([t]_2, [u]_2) \leftarrow_{\$} \text{Tag}(\text{sk}_{\text{MAC}}, m)$	<u>CHAL(m<sup>*</sup>):</u> //one query $h \leftarrow_{\$} \mathbb{Z}_q^*$ $\mathbf{h}_0 = \sum f_i(m^*) \mathbf{x}_i \cdot h \in \mathbb{Z}_q^n; h_1 = \sum f'_i(m^*) x'_i \cdot h \in \mathbb{Z}_q$ $[\mathbf{h}_0] \leftarrow_{\$} \mathbb{Z}_q^n; [h_1] \leftarrow_{\$} \mathbb{Z}_q$ Return $([h]_1, [\mathbf{h}_0]_1, [h_1]_T)$  <u>FINALIZE(d ∈ {0, 1}):</u> Return $d \wedge (m^* \notin \mathcal{Q}_{\mathcal{M}})$
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**Figure 1:** Games  $\text{PR-CMA}_{\text{real}}$  and  $\text{PR-CMA}_{\text{rand}}$  for defining PR-CMA security. In all procedures, the boxed statements redefining  $(\mathbf{h}_0, h_1)$  are only executed in game  $\text{PR-CMA}_{\text{rand}}$ .

Let  $\mathcal{G} = (\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, q, g_1, g_2, e)$  be an asymmetric pairing group such that  $(\mathbb{G}_2, g_2, q)$  is contained in  $\text{par}$ . We define the PR-CMA security via games  $\text{PR-CMA}_{\text{real}}$  and  $\text{PR-CMA}_{\text{rand}}$  from Figure 1. Note that the output  $([h]_1, [\mathbf{h}_0]_1, [h_1]_T)$  of  $\text{CHAL}(m^*)$  in game  $\text{PR-CMA}_{\text{real}}$  can be viewed as a “token” for message  $m^*$  to check verification equation (3) for arbitrary tags  $([t]_2, [u]_2)$  via equation  $e([h]_1, [u]_2) \stackrel{?}{=} e([t]_1, [\mathbf{h}_0]_1) \cdot [h_1]_T$ . Intuitively, the pseudorandomness of  $[h_1]_T$  is responsible for indistinguishability and of  $[\mathbf{h}_0]_1$  to prove anonymity of the IBE scheme.

**Definition 3.2** An affine MAC over  $\mathbb{Z}_q^n$  is PR-CMA-secure if for all PPT  $\mathcal{A}$ ,  $\text{Adv}_{\text{MAC}}^{\text{pr-cma}}(\mathcal{A}) := \Pr[\text{PR-CMA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] - \Pr[\text{PR-CMA}_{\text{rand}}^{\mathcal{A}} \Rightarrow 1]$  is negligible, where the experiments are defined in Figure 1.

### 3.2 An Affine MAC from the Naor-Reingold PRF

Unfortunately, the (deterministic) Naor-Reingold pseudorandom function is not affine. We use the following randomized version  $\text{MAC}_{\text{NR}}[\mathcal{D}_k] = (\text{Gen}_{\text{MAC}}, \text{Tag}, \text{Ver})$  of it based on any matrix assumption  $\mathcal{D}_k$ . For the special case  $\mathcal{D}_k = \mathcal{L}_k$ , it was implicitly given in [7]. Recall that, for any matrix  $\mathbf{A} \in \mathbb{Z}_q^{(k+1) \times k}$  we denote the upper  $k$  rows by  $\bar{\mathbf{A}} \in \mathbb{Z}_q^{k \times k}$  and the last row by  $\underline{\mathbf{A}} \in \mathbb{Z}_q^{1 \times k}$ .

<u>Gen<sub>MAC</sub>(par):</u> $\mathbf{A} \leftarrow \mathcal{D}_k; \mathbf{B} := \bar{\mathbf{A}} \in \mathbb{Z}_q^{k \times k}$ $\mathbf{x}_{1,0}, \dots, \mathbf{x}_{m,1} \leftarrow_{\$} \mathbb{Z}_q^k; x'_0 \leftarrow_{\$} \mathbb{Z}_q$ Return $\text{sk}_{\text{MAC}} = (\mathbf{B}, \mathbf{x}_{1,0}, \dots, \mathbf{x}_{m,1}, x'_0)$	<u>Tag(sk<sub>MAC</sub>, m):</u> $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s}$ $u = (\sum_{i=1}^{ m } \mathbf{x}_{i,m_i}^\top) \mathbf{t} + x'_0 \in \mathbb{Z}_q$ Return $\tau = ([t]_2, [u]_2) \in \mathbb{G}_2^k \times \mathbb{G}_2$	<u>Ver(sk<sub>MAC</sub>, τ, m):</u> If $u = (\sum_{i=1}^{ m } \mathbf{x}_{i,m_i}^\top) \mathbf{t} + x'_0$ then return 1; Else return 0.
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Note that  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$  is  $n$ -affine over  $\mathbb{Z}_q^n$  with message space  $\mathcal{M} = \{0, 1\}^m$ . Writing  $\mathbf{x}_{i,b} = \mathbf{x}_{2i+b}$  we have  $n = n' = k, \ell' = 0, \ell = 2m + 1$  and functions  $f_0(m) = f_1(m) = 0, f'_0(m) = 1$ , and  $f_{2i+b}(m) = (m_i = b)$  for  $1 \leq i \leq m$ . (To perfectly fit our definition,  $\mathbf{x}_{i,b}$  should be renamed to  $\mathbf{x}_{2i+b}$ , but we conserve the other notations for better readability.)

**Theorem 3.3**  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$  is tightly PR-CMA-secure under the  $\mathcal{D}_k$ -MDDH assumption. In particular, for all adversaries  $\mathcal{A}$  there exists an adversary  $\mathcal{D}$  with  $\mathbf{T}(\mathcal{A}) \approx \mathbf{T}(\mathcal{D})$  and  $\text{Adv}_{\text{MAC}_{\text{NR}}[\mathcal{D}_k]}^{\text{pr-cma}}(\mathcal{A}) \leq 4m(\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{D}) - 1/(q - 1))$ .

Note that the security bound is (almost) tight, as  $m$  is the bit-length of message space  $\mathcal{M}$ . The proof follows the ideas from [7, 24]. We use  $m$  hybrids, where in hybrid  $i$  all the (maximal  $Q$ ) values  $\mathbf{x}_{i,1-m_i^*}^\top \cdot \mathbf{t}$  in the response to an EVAL query are replaced by uniform randomness. Here  $m^*$  is the message from the challenge query. We use the  $Q$ -fold  $\mathcal{D}_k$ -MDDH assumption to interpolate between the hybrids, where the reductions guesses  $m_i^*$  correctly with probability  $1/2$ . As the  $Q$ -fold  $\mathcal{D}_k$ -MDDH assumption is tightly implied by the standard  $\mathcal{D}_k$ -MDDH assumption (Lemma 2.3), the proof follows.

We remark, that one can define an alternative version of  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$  by setting  $\mathbf{x}_0 := \sum \mathbf{x}_{i,0}, \mathbf{x}_i := \mathbf{x}_{i,1} - \mathbf{x}_{i,0}$  and  $u = (\mathbf{x}_0^\top + \sum_{i=1}^{|m|} m_i \mathbf{x}_i^\top) \mathbf{t} + x'_0$ . This MAC has a shorter secret key and can also be shown to be PR-CMA. (However, it does not satisfy the stronger security notion of HPR-CMA needed in Section 5.)



<b>INITIALIZE:</b> // Games $G_0, \boxed{G_{1,i}}, G_2$ For $j = 1, \dots, m : \mathbf{x}_{j,0}, \mathbf{x}_{j,1} \leftarrow \mathbb{Z}_q^k$ $x'_0 \leftarrow_{\$} \mathbb{Z}_q$ ; $x'_0$ is undefined Return $\varepsilon$  <b>CHAL(<math>\mathbf{m}^*</math>):</b> // Games $G_0, \boxed{G_{1,i}}, \boxed{G_{2,i}}$ , one query $h \leftarrow_{\$} \mathbb{Z}_q$ ; $x'_0 = \text{RF}_i(\mathbf{m}_{ i}^*)$ $\mathbf{h}_0 = (\sum_{j=1}^m \mathbf{x}_{j,m_j^*}) \cdot h$ ; $h_1 = x'_0 \cdot h \in \mathbb{Z}_q$ $\boxed{h_0 \leftarrow_{\$} \mathbb{Z}_q^k; h_1 \leftarrow_{\$} \mathbb{Z}_q}$ Return $([h]_1, [\mathbf{h}_0]_1, [h_1]_T)$	<b>EVAL(<math>\mathbf{m}</math>):</b> // Games $G_0, \boxed{G_{1,i}}, \boxed{G_{2,i}}$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ $\mathbf{t} \leftarrow_{\$} \mathbb{Z}_q^k$ $u = \sum_{j=1}^m \mathbf{x}_{j,m_j}^\top \mathbf{t} + x'_0$ $\boxed{u = \sum_{j=1}^m \mathbf{x}_{j,m_j}^\top \mathbf{t} + \text{RF}_i(\mathbf{m}_{ i})}$ $\boxed{u \leftarrow_{\$} \mathbb{Z}_q}$ Return $([\mathbf{t}]_2, [u]_2)$  <b>FINALIZE(<math>d \in \{0, 1\}</math>):</b> // Games $G_0$ - $G_2$ Return $d \wedge (\mathbf{m}^* \notin \mathcal{Q}_{\mathcal{M}})$
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**Figure 2:** Games  $G_0, G_{1,i}$  ( $0 \leq i \leq m$ ) and  $G_2$  for the proof of Lemmas 3.4 to 3.6.  $\text{RF}_i : \{0, 1\}^i \rightarrow \mathbb{Z}_q$  is a random function and  $\mathbf{m}_{|i}$  denotes the  $i$ th prefix of  $\mathbf{m}$ . In each procedure, a solid (dotted) frame indicates that the command is only executed in the game marked by a solid (dotted) frame.

**Proof of Theorem 3.3:** Let  $\mathcal{A}$  be an adversary against the PR-CMA-security of  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$ . We prove Theorem 3.3 by defining a sequence of intermediate games as in Figure 2 and 4.

Game  $G_0$  is the real attack game and in  $G_{1,0}$ , we syntactically replace  $x'_0$  by  $\text{RF}_0(\epsilon)$  which is a fixed random element.

**Lemma 3.4**  $\Pr[\text{PR-CMA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] = \Pr[G_0^{\mathcal{A}} \Rightarrow 1] = \Pr[G_{1,0}^{\mathcal{A}} \Rightarrow 1]$ .

**Lemma 3.5** *There exists an adversary  $\mathcal{B}$  with  $\mathbf{T}(\mathcal{B}) \approx \mathbf{T}(\mathcal{A})$  and  $|\Pr[G_{1,i}^{\mathcal{A}} \Rightarrow 1] - \Pr[G_{1,i-1}^{\mathcal{A}} \Rightarrow 1]| \leq 2(\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}) - \frac{1}{q-1})$*

**Proof:** Let  $Q$  be the maximal number of EVAL queries made by  $\mathcal{A}$ . We first build an adversary  $\mathcal{B}'$  against the  $Q$ -fold  $\mathcal{D}_k$ -MDDH Assumption such that

$$2\text{Adv}_{\mathcal{D}_k, \text{GGen}}^Q(\mathcal{B}') \geq |\Pr[G_{1,i}^{\mathcal{A}} \Rightarrow 1] - \Pr[G_{1,i-1}^{\mathcal{A}} \Rightarrow 1]|. \quad (4)$$

This implies the lemma using the random self reducibility of the MDDH assumption (Lemma 2.3).

On input of a  $Q$ -fold  $\mathcal{D}_k$ -MDDH challenge  $([\mathbf{A}]_2, [\mathbf{H}]_2) \in \mathbb{G}_2^{(k+1) \times k} \times \mathbb{G}_2^{(k+1) \times Q}$ ,  $\mathcal{B}'$  first picks a random bit  $b$  which is a guess for  $\mathbf{m}_i^*$ . Let  $\text{RF}_{i-1}$  and  $\text{RF}'_{i-1}$  be two independent random functions (defined on the fly) which we use to define

$$\text{RF}_i(\mathbf{m}_{|i}) = \begin{cases} \text{RF}_{i-1}(\mathbf{m}_{|i-1}) & \mathbf{m}_i = b \\ \text{RF}_{i-1}(\mathbf{m}_{|i-1}) + \text{RF}'_{i-1}(\mathbf{m}_{|i-1}) & \mathbf{m}_i = 1 - b \end{cases},$$

The construction of  $\mathcal{B}'$  is described in Figure 3. Note that if  $\text{RF}_{i-1}$  and  $\text{RF}'_{i-1}$  are random functions, then  $\text{RF}_i$  is a random function.

Assume  $\mathcal{B}'$  correctly guesses  $b = \mathbf{m}_i^*$  (which happens with probability  $1/2$ ). By the definition of  $\text{RF}_i$  and by  $\mathbf{m}_i^* = b$  we have  $\text{RF}_i(\mathbf{m}_{|i}^*) = \text{RF}_{i-1}(\mathbf{m}_{|i-1}^*)$ , which implies  $\text{CHAL}(\mathbf{m}^*)$  is identically distributed in  $G_{1,i}$  and  $G_{1,i-1}$ .

We now analyze the output distribution of the EVAL queries. First note that  $\mathbf{t}$  is uniformly random over  $\mathbb{Z}_q^k$  in both games  $G_{1,i}$  and  $G_{1,i-1}$ . As for the distribution of  $u$ , we only need to consider the case  $\mathbf{m}_i = 1 - b$ , since  $u$  for  $\mathbf{m}_i = b$  is identically distributed in games  $G_{1,i}$  and  $G_{1,i-1}$ . Assume  $\mathbf{m}_i = 1 - b$ . Write  $\mathbf{H}_c = \mathbf{A}\mathbf{W}_c + \mathbf{R}_c$  for some  $\mathbf{W}_c \in \mathbb{Z}_q^k$ , where  $\mathbf{R}_c = 0$  (i.e.,  $\mathbf{H}$  is from the  $\mathcal{D}_k$ -MDDH distribution) or

<p><u>INITIALIZE:</u>  <math>b \leftarrow_{\mathcal{S}} \{0, 1\}</math>  For <math>j = 1, \dots, m</math> and <math>j' = 0, 1</math> :      If <math>j \neq i</math> or <math>j' = b</math> then <math>\mathbf{x}_{j,j'} \leftarrow_{\mathcal{S}} \mathbb{Z}_q^k</math>  <math>\mathbf{r} \leftarrow_{\mathcal{S}} \mathbb{Z}_q^{k+1}</math>; <math>\mathbf{x}_{i,1-b}^{\top} \bar{\mathbf{A}} := \mathbf{r}^{\top} \mathbf{A} \in \mathbb{Z}_q^k</math>  Return <math>\varepsilon</math></p> <p><u>CHAL(<math>\mathbf{m}^*</math>):</u>  Abort if <math>\mathbf{m}_i^* \neq b</math>.  <math>h \leftarrow_{\mathcal{S}} \mathbb{Z}_q</math>; <math>x'_0 = \text{RF}_{i-1}(\mathbf{m}_{ i-1}^*)</math>  <math>\mathbf{h}_0 = (\sum_{j=1}^m \mathbf{x}_{j,\mathbf{m}_j^*})h \in \mathbb{Z}_q^k</math>  <math>h_1 = x'_0 h \in \mathbb{Z}_q</math>  Return <math>([h]_1, [\mathbf{h}_0]_1, [h_1]_T)</math></p>	<p><u>EVAL(<math>\mathbf{m}</math>):</u>  <math>\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}</math>  <math>c := \alpha_i(\mathbf{m}_{ i})</math>  <math>\mathbf{s}' \leftarrow_{\mathcal{S}} \mathbb{Z}_q^k</math>; <math>\mathbf{t} = \bar{\mathbf{A}}\mathbf{s}' + \bar{\mathbf{H}}_c</math>  <math>u = \begin{cases} (\sum_{j=1}^{ \mathbf{m} } \mathbf{x}_{j,\mathbf{m}_j}^{\top})\mathbf{t} + \text{RF}_{i-1}(\mathbf{m}_{ i-1}) &amp; \mathbf{m}_i = b \\ (\sum_{j \neq i} \mathbf{x}_{j,\mathbf{m}_j}^{\top})\mathbf{t} + \mathbf{r}^{\top}(\mathbf{A}\mathbf{s}' + \mathbf{H}_c) + \text{RF}_{i-1}(\mathbf{m}_{ i-1}) &amp; \mathbf{m}_i = 1 - b \end{cases}</math>  Return <math>([\mathbf{t}]_2, [u]_2)</math></p> <p><u>FINALIZE(<math>d \in \{0, 1\}</math>):</u>  Return <math>d \wedge (\mathbf{m}^* \notin \mathcal{Q}_{\mathcal{M}})</math></p>
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**Figure 3:** Description of  $\mathcal{B}'(\mathcal{G}, [\mathbf{A}]_2, [\mathbf{H}]_2)$  interpolating between the Games  $\mathbf{G}_{1,i}$  and  $\mathbf{G}_{1,i-1}$ , where  $\mathbf{H}_c$  denotes the  $c$ -th column of  $\mathbf{H}$  and  $\alpha_i : \{0, 1\}^i \rightarrow \{1, \dots, Q\}$  is an injective function.

$\mathbf{R}_c$  is uniform. Then,

$$\begin{aligned}
u &= \sum_{j \neq i} \mathbf{x}_{j,\mathbf{m}_j}^{\top} \mathbf{t} + \mathbf{r}^{\top} \mathbf{A}(\mathbf{s}' + \mathbf{W}_c) + \mathbf{r}^{\top} \mathbf{R}_c + \text{RF}_{i-1}(\mathbf{m}_{|i-1}) \\
&= \sum_{j \neq i} \mathbf{x}_{j,\mathbf{m}_j}^{\top} \mathbf{t} + \mathbf{x}_{i,1-b}^{\top} \underbrace{\bar{\mathbf{A}}(\mathbf{s}' + \mathbf{W}_c)}_{\mathbf{t}} + \mathbf{r}^{\top} \mathbf{R}_c + \text{RF}_{i-1}(\mathbf{m}_{|i-1}) \\
&= \sum_{j=0}^{|\mathbf{m}|} \mathbf{x}_{j,\mathbf{m}_j}^{\top} \mathbf{t} + \mathbf{r}^{\top} \mathbf{R}_c + \text{RF}_{i-1}(\mathbf{m}_{|i-1}).
\end{aligned}$$

If  $\mathbf{R}_c = 0$ , then  $u$  is distributed as in game  $\mathbf{G}_{1,i-1}$ . If  $\mathbf{R}_c$  is uniform, then define  $\text{RF}'(\mathbf{m}_{|i-1}) := \mathbf{r}^{\top} \mathbf{R}_c$  and  $u$  is distributed as in  $\mathbf{G}_{1,i}$ .  $\blacksquare$

**Lemma 3.6**  $\Pr[\mathbf{G}_{1,m}^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1]$

**Proof:** In  $\mathbf{G}_{1,m}$ , the values  $u$  computed in  $\text{EVAL}(\mathbf{m})$  are masked by  $\text{RF}_m(\mathbf{m})$  and are hence uniformly random.  $\blacksquare$

Finally, we do all the previous steps in reverse order, as shown in Figure 4. Clearly,  $\mathbf{H}_2 = \mathbf{G}_2$  and  $\mathbf{H}_0 = \text{PR-CMA}_{\text{rand}}$ . Following the arguments of Lemmas 3.4 to 3.6 in reverse order, one obtains the following lemma.

**Lemma 3.7** *There exists an adversary  $\mathcal{B}$  with  $\mathbf{T}(\mathcal{B}) \approx \mathbf{T}(\mathcal{A})$  and  $2m(\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}) - 1/(q-1)) \geq |\Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1] - \Pr[\text{PR-CMA}_{\text{rand}}^{\mathcal{A}} \Rightarrow 1]|$ .*

This completes the proof of Theorem 3.3.  $\blacksquare$

### 3.3 An Affine MAC from Hash Proof System

Let  $\mathcal{D}_k$  be a matrix distribution. We now combine the hash proof system for the subset membership problem induced by the  $\mathcal{D}_k$ -MDDH assumption from [12] with the generic MAC construction from [11] and obtain the following  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  for  $\mathcal{M} = \mathbb{Z}_q^\ell$ .

<b>INITIALIZE:</b> // Games $H_0, H_{1,i}, \overline{H_2}$ $\mathbf{B} \leftarrow_{\$} \mathbb{Z}_q^{k \times k}; \overline{x_0} \leftarrow_{\$} \mathbb{Z}_q$ For $i = 1, \dots, m : \mathbf{x}_{i,0}, \mathbf{x}_{i,1} \leftarrow_{\$} \mathbb{Z}_q^k$ Return $\varepsilon$  <b>CHAL(<math>\mathbf{m}^*</math>):</b> // Games $H_0$ - $H_2$ , one query $h \leftarrow_{\$} \mathbb{Z}_q$ $\mathbf{h}_0 \leftarrow_{\$} \mathbb{Z}_q^k; h_1 \leftarrow_{\$} \mathbb{Z}_q$ Return $([h]_1, [\mathbf{h}_0]_1, [h_1]_T)$	<b>EVAL(<math>\mathbf{m}</math>):</b> // Games $H_0, \overline{H_{1,i}}, \overline{H_2}$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s}$ $u \leftarrow_{\$} \mathbb{Z}_q$ $u = (\sum_{j=1}^m \mathbf{x}_{j,m_j}^\top) \mathbf{t} + \text{RF}_i(\mathbf{m}_{ i})$ $u = (\sum_{j=1}^m \mathbf{x}_{j,m_j}^\top) \mathbf{t} + x'_0$ Return $([\mathbf{t}]_2, [u]_2)$  <b>FINALIZE(<math>d \in \{0, 1\}</math>):</b> // Games $H_0$ - $H_2$ Return $d \wedge (\mathbf{m}^* \notin \mathcal{Q}_{\mathcal{M}})$
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**Figure 4:** Games  $H_0, H_{1,i}$  ( $0 \leq i \leq m$ ) and  $H_2$  for the proof of Lemma 3.7.

<b>Gen<sub>MAC</sub>(par):</b> $\mathbf{B} \leftarrow_{\$} \mathcal{D}_k$ $\mathbf{x}_0, \dots, \mathbf{x}_\ell \leftarrow_{\$} \mathbb{Z}_q^{k+1}$ $x'_0 \leftarrow_{\$} \mathbb{Z}_q$ Return $\text{sk}_{\text{MAC}} = (\mathbf{B}, \mathbf{x}_0, \dots, \mathbf{x}_\ell, x'_0)$	<b>Tag(sk<sub>MAC</sub>, m):</b> $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k$ $\mathbf{t} = \mathbf{B}\mathbf{s} \in \mathbb{Z}_q^{k+1}$ $u = (\mathbf{x}_0^\top + \sum_{i=1}^{ \mathbf{m} } \mathbf{m}_i \cdot \mathbf{x}_i^\top) \mathbf{t} + x'_0 \in \mathbb{Z}_q$ Return $\tau = ([\mathbf{t}]_2, [u]_2) \in \mathbb{G}_2^{k+1} \times \mathbb{G}_2$	<b>Ver(sk<sub>MAC</sub>, <math>\tau, \mathbf{m}</math>):</b> If $u = (\mathbf{x}_0^\top + \sum_{i=1}^{ \mathbf{m} } \mathbf{m}_i \cdot \mathbf{x}_i^\top) \mathbf{t} + x'_0$ then return 1 Else return 0
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Note that  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  is  $n$ -affine over  $\mathbb{Z}_q^n$  with  $n = k+1$ ,  $n' = k$ ,  $\ell' = 0$ , and defining functions  $f_0(\mathbf{m}) = 1$ ,  $f_i(\mathbf{m}) = \mathbf{m}_i$ , and  $f'_0(\mathbf{m}) = 1$ , where  $\mathbf{m}_i$  is the  $i$ -th component of  $\mathbf{m}$ . For the moment we use  $\ell = 1$  which already gives a MAC with exponential message space  $\mathcal{M} = \mathbb{Z}_q$ .

Combining [12, 11] we obtain that  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  is UF-CMA under the  $\mathcal{D}_k$ -MDDH assumption. The proof extends to show even PR-CMA security. Compared to  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$ , we lose the tight reduction, but gain much shorter public parameters.

**Theorem 3.8**  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  is PR-CMA-secure under the  $\mathcal{D}_k$ -MDDH assumption. In particular, for all adversaries  $\mathcal{A}$  there exists an adversary  $\mathcal{D}$  with  $\mathbf{T}(\mathcal{A}) \approx \mathbf{T}(\mathcal{D})$  and  $\text{Adv}_{\text{MAC}_{\text{HPS}}[\mathcal{D}_k]}^{\text{pr-cma}}(\mathcal{A}) \leq 2Q(\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{D}) + 1/q)$ , where  $Q$  is the maximal number of queries to  $\text{EVAL}(\cdot)$ .

<b>INITIALIZE:</b> // Games $G_0$ - $G_2$ $\mathbf{B} \leftarrow_{\$} \mathcal{D}_k; x'_0 \leftarrow_{\$} \mathbb{Z}_q$ For $j = 0, \dots, \ell : \mathbf{x}_j \leftarrow_{\$} \mathbb{Z}_q^{k+1}$ Return $\varepsilon$  <b>CHAL(<math>\mathbf{m}^*</math>):</b> // Games $G_0$ - $G_{1,Q+1}, \overline{G_2}$ $h \leftarrow_{\$} \mathbb{Z}_q$ $\mathbf{h}_0 = (\mathbf{x}_0 + \sum \mathbf{m}_j^* \cdot \mathbf{x}_j) h \in \mathbb{Z}_q^{k+1}; h_1 = x'_0 h \in \mathbb{Z}_q$ $\overline{\mathbf{h}_0} \leftarrow_{\$} \mathbb{Z}_q^{k+1}; \overline{h_1} \leftarrow_{\$} \mathbb{Z}_q$ Return $([h]_1, [\mathbf{h}_0]_1, [h_1]_T)$  <b>FINALIZE(<math>d \in \{0, 1\}</math>):</b> // Games $G_0$ - $G_2$ Return $d \wedge (\mathbf{m}^* \notin \mathcal{Q}_{\mathcal{M}})$  <b>EVAL(<math>\mathbf{m}</math>):</b> // Game $G_2$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ $(\mathbf{t}, u) \leftarrow_{\$} \mathbb{Z}_q^{k+1} \times \mathbb{Z}_q$ Return $([\mathbf{t}]_2, [u]_2)$	<b>EVAL(<math>\mathbf{m}</math>):</b> // Game $G_0$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s} \in \mathbb{Z}_q^{k+1}$ $u = (\mathbf{x}_0^\top + \sum_{j=1}^m \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0$ Return $([\mathbf{t}]_2, [u]_2)$  <b>EVAL(<math>\mathbf{m}</math>):</b> // Games $G_{1,i}, \overline{G'_{1,i}}$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ // Let $\mathbf{m}$ be the $c$ -th query $(1 \leq c \leq Q)$ If $c < i$ then $(\mathbf{t}, u) \leftarrow_{\$} \mathbb{Z}_q^{k+1} \times \mathbb{Z}_q$ If $c > i$ then $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s}; u = (\mathbf{x}_0^\top + \sum_{j=1}^{ \mathbf{m} } \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0$ If $c = i$ then $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s}$ $\overline{\mathbf{t}} \leftarrow_{\$} \mathbb{Z}_q^{k+1}$ $u = (\mathbf{x}_0^\top + \sum_{j=1}^{ \mathbf{m} } \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0$ Return $([\mathbf{t}]_2, [u]_2)$
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**Figure 5:** Games  $G_0, (G_{1,i}, G'_{1,i})_{1 \leq i \leq Q}, G_{1,Q+1}, G_2$  for the proof of Theorem 3.8.

**Proof:** We prove Theorem 3.8 by defining a sequence of intermediate games as in Figure 5. Let  $\mathcal{A}$  be an adversary against the PR-CMA-security of  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$ . Game  $G_0$  is the real attack game. In games  $G_{1,i}$ , the first  $i-1$  queries to the EVAL oracle are answered with uniform values in  $\mathbb{G}_2^{k+1} \times \mathbb{G}_2$  and the rest are answered as in the real scheme. To interpolate between  $G_{1,i}$  and  $G_{1,i+1}$ , we also define  $G'_{1,i}$ , which answers the  $i$ -th query to EVAL by picking a random  $\mathbf{t} \leftarrow_s \mathbb{Z}_q^{k+1}$ . By definition, we have  $G_0 = G_{1,1}$ .

**Lemma 3.9**  $\Pr[\text{PR-CMA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] = \Pr[G_0^{\mathcal{A}} \Rightarrow 1] = \Pr[G_{1,1}^{\mathcal{A}} \Rightarrow 1]$ .

**Lemma 3.10** *There exists an adversary  $\mathcal{B}$  with  $\mathbf{T}(\mathcal{B}) \approx \mathbf{T}(\mathcal{A})$  and  $\mathbf{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}) \geq |\Pr[G_{1,i}^{\mathcal{A}} \Rightarrow 1] - \Pr[G'_{1,i}^{\mathcal{A}} \Rightarrow 1]|$ .*

**Proof:** Games  $G_{1,i}$  and  $G'_{1,i}$  only differ in the distribution of  $\mathbf{t}$  returned by the EVAL oracle for its  $i$ -th query, namely,  $\mathbf{t} \in \text{span}(\mathbf{B})$  or uniform. From that, we obtain a straightforward reduction to the  $\mathcal{D}_k$ -MDDH Assumption.  $\blacksquare$

**Lemma 3.11**  $|\Pr[G_{1,i+1}^{\mathcal{A}} \Rightarrow 1] - \Pr[G'_{1,i}^{\mathcal{A}} \Rightarrow 1]| \leq 1/q$ .

**Proof:** At a high level, these two games are only separated by the 2-universality of the underlying hash proof system. Let  $\mathbf{m}$  be the  $i$ -th query to EVAL and let  $([\mathbf{t}]_2, [u]_2)$  be its tag. As  $\mathbf{m} \neq \mathbf{m}^*$ , there exists an index  $i'$  such that  $\mathbf{m}_{i'} \neq \mathbf{m}_{i'}^*$ , where  $\mathbf{m}_{i'}$  (resp.  $\mathbf{m}_{i'}^*$ ) denotes the  $i'$ -th entry of  $\mathbf{m}$  (resp.  $\mathbf{m}^*$ ). We use an information-theoretic argument to show that in  $G'_{1,i}$  the value  $u - x'_0$  is uniformly random. For simplicity, we assume  $x'_0$  and  $\mathbf{x}_j$  ( $j \notin \{0, i'\}$ ) are known to  $\mathcal{A}$ . Information-theoretically, adversary  $\mathcal{A}$  may also learn  $\mathbf{B}^\top \mathbf{x}_0$  and  $\mathbf{B}^\top \mathbf{x}_{i'}$  from the  $c$ -th query with  $c > i$ . Thus,  $\mathcal{A}$  information-theoretically obtains the following equations in the unknown variables  $\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_{i'} \end{pmatrix} \in \mathbb{Z}_q^{2(k+1)}$ :

$$\begin{pmatrix} \mathbf{B}^\top \mathbf{x}_0 \\ \mathbf{B}^\top \mathbf{x}_{i'} \\ \mathbf{h}_0 \\ u - x'_0 \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{B}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^\top \\ h \cdot \mathbf{I}_{k+1} & \mathbf{m}_{i'}^* h \cdot \mathbf{I}_{k+1} \\ \mathbf{t}^\top & \mathbf{m}_{i'} \mathbf{t}^\top \end{pmatrix}}_{=: \mathbf{M} \in \mathbb{Z}_q^{(3k+2) \times (2k+2)}} \cdot \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_{i'} \end{pmatrix}$$

where  $\mathbf{I}_{k+1}$  is the  $(k+1) \times (k+1)$  identity matrix. To show that  $u - x'_0$  is linearly independent of  $\mathbf{B}^\top \mathbf{x}_0$ ,  $\mathbf{B}^\top \mathbf{x}_{i'}$  and  $\mathbf{h}_0$ , we argue that the last row of  $\mathbf{M}$  is linearly independent of all the other rows. Since  $\mathbf{t} \notin \text{span}(\mathbf{B})$  (except with probability  $1/q$ ),  $\mathbf{t}^\top$  is independent of  $\mathbf{B}^\top$ ; by  $\mathbf{m}_{i'} \neq \mathbf{m}_{i'}^*$ , the last row of  $\mathbf{M}$  is linearly independent of rows  $2k+1$  to  $3k+1$ . We conclude that  $u$  is uniformly random in  $\mathcal{A}$ 's view.  $\blacksquare$

**Lemma 3.12**  $\Pr[G_2^{\mathcal{A}} \Rightarrow 1] = \Pr[G_{1,Q+1}^{\mathcal{A}} \Rightarrow 1]$ .

**Proof:** Note that  $\mathcal{A}$  can ask at most  $Q$ -many EVAL queries. In both  $G_{1,Q+1}$  and  $G_2$ , all answers of EVAL are uniformly at random and independent of the secret keys  $(x'_0, \mathbf{x}_0, \dots, \mathbf{x}_\ell)$ . Hence, the values  $\mathbf{h}_0$  and  $h_1$  from  $G_{1,Q+1}$  are uniform in the view of  $\mathcal{A}$ .  $\blacksquare$

We now do all the previous steps in the reverse order as in Figure 6. Then, by using the above arguments in a reverse order, we have the following lemma.

**Lemma 3.13** *There exists an adversary  $\mathcal{B}$  with  $\mathbf{T}(\mathcal{B}) \approx \mathbf{T}(\mathcal{A})$  and  $Q(\mathbf{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}) + 1/q) \geq |\Pr[\text{PR-CMA}_{\text{rand}}^{\mathcal{A}} \Rightarrow 1] - \Pr[G_2^{\mathcal{A}} \Rightarrow 1]|$ .*

Theorem 3.8 follows by combining Lemmas 3.9-3.13.  $\blacksquare$

<b>INITIALIZE:</b> // Games $H_0$ - $H_2$ $\mathbf{B} \leftarrow_{\$} \mathcal{D}_k$ ; $x'_0 \leftarrow_{\$} \mathbb{Z}_q$ For $j = 0, \dots, \ell$ : $\mathbf{x}_j \leftarrow_{\$} \mathbb{Z}_q^{k+1}$ Return $\varepsilon$  <b>CHAL(<math>\mathbf{m}^*</math>):</b> // Games $H_0$ - $H_2$ $h \leftarrow_{\$} \mathbb{Z}_q$ ; $\mathbf{h}_0 \leftarrow_{\$} \mathbb{Z}_q^{k+1}$ ; $h_1 \leftarrow_{\$} \mathbb{Z}_q$ Return $([h]_1, [\mathbf{h}_0]_1, [h_1]_T)$  <b>FINALIZE(<math>d \in \{0, 1\}</math>):</b> // Games $H_0$ - $H_2$ Return $d \wedge (\mathbf{m}^* \notin \mathcal{Q}_{\mathcal{M}})$  <b>EVAL(<math>\mathbf{m}</math>):</b> // Game $H_0$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ $(\mathbf{t}, u) \leftarrow_{\$} \mathbb{Z}_q^{k+1} \times \mathbb{Z}_q$ Return $([\mathbf{t}]_2, [u]_2)$	<b>EVAL(<math>\mathbf{m}</math>):</b> // Games $H_{1,i}, \boxed{H'_{1,i}}$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ // Let $\mathbf{m}$ be the $c$ -th query ( $1 \leq c \leq Q$ ) If $c > i$ then $(\mathbf{t}, u) \leftarrow_{\$} \mathbb{Z}_q^{k+1} \times \mathbb{Z}_q$ If $c < i$ then $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k$ , $\mathbf{t} = \mathbf{B}\mathbf{s}$ ; $u = (\mathbf{x}_0^\top + \sum_{j=1}^{ \mathbf{m} } \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0$ If $c = i$ then $\mathbf{t} \leftarrow_{\$} \mathbb{Z}_q^{k+1}$ <div style="border: 1px solid black; padding: 2px; display: inline-block;"><math>\mathbf{s} \leftarrow_{\\$} \mathbb{Z}_q^k</math>; <math>\mathbf{t} = \mathbf{B}\mathbf{s} \in \mathbb{Z}_q^{k+1}</math></div> $u = (\mathbf{x}_0^\top + \sum_{j=1}^{ \mathbf{m} } \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0$ Return $([\mathbf{t}]_2, [u]_2)$  <b>EVAL(<math>\mathbf{m}</math>):</b> // Game $H_2$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k$ , $\mathbf{t} = \mathbf{B}\mathbf{s}$ $u = (\mathbf{x}_0^\top + \sum_{i=1}^{ \mathbf{m} } \mathbf{m}_i \cdot \mathbf{x}_i^\top) \mathbf{t} + x'_0$ Return $([\mathbf{t}]_2, [u]_2)$
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**Figure 6:** Games  $H_0, (H_{1,i}, H'_{1,i})_{1 \leq i \leq Q}, H_{1,Q+1}, H_2$  for the proof of Lemma 3.13.

## 4 Identity-based Encryption from Affine MACs

In this section, we will present our transformation  $\text{IBE}[\text{MAC}, \mathcal{D}_k]$  from affine MACs to IBE based on the  $\mathcal{D}_k$ -MDDH assumption.

### 4.1 Identity-based Key Encapsulation

We now recall syntax and security of IBE in terms of an ID-based key encapsulation mechanism IBKEM. Every IBKEM can be transformed into an ID-based encryption scheme IBE using a (one-time secure) symmetric cipher.

**Definition 4.1 (Identity-based Key Encapsulation Scheme)** *An identity-based key encapsulation (IBKEM) scheme IBKEM consists of four PPT algorithms  $\text{IBKEM} = (\text{Gen}, \text{USKGen}, \text{Enc}, \text{Dec})$  with the following properties.*

- The probabilistic key generation algorithm  $\text{Gen}(1^\lambda)$  returns the (master) public/secret key  $(\mathbf{pk}, \mathbf{sk})$ . We assume that  $\mathbf{pk}$  implicitly defines a message space  $\mathcal{M}$ , an identity space  $\mathcal{ID}$ , a key space  $\mathcal{K}$ , and ciphertext space  $\mathcal{C}$ .
- The probabilistic user secret key generation algorithm  $\text{USKGen}(\mathbf{sk}, \text{id})$  returns the user secret-key  $\text{usk}[\text{id}]$  for identity  $\text{id} \in \mathcal{ID}$ .
- The probabilistic encapsulation algorithm  $\text{Enc}(\mathbf{pk}, \text{id})$  returns the symmetric key  $\mathbf{K} \in \mathcal{K}$  together with a ciphertext  $\mathbf{C} \in \mathcal{C}$  with respect to identity  $\text{id}$ .
- The deterministic decapsulation algorithm  $\text{Dec}(\text{usk}[\text{id}], \text{id}, \mathbf{C})$  returns the decapsulated key  $\mathbf{K} \in \mathcal{K}$  or the reject symbol  $\perp$ .

For perfect correctness we require that for all  $\lambda \in \mathbb{N}$ , all pairs  $(\mathbf{pk}, \mathbf{sk})$  generated by  $\text{Gen}(1^\lambda)$ , all identities  $\text{id} \in \mathcal{ID}$ , all  $\text{usk}[\text{id}]$  generated by  $\text{USKGen}(\mathbf{sk}, \text{id})$  and all  $(\mathbf{K}, \mathbf{C})$  output by  $\text{Enc}(\mathbf{pk}, \text{id})$ :

$$\Pr[\text{Dec}(\text{usk}[\text{id}], \text{id}, \mathbf{C}) = \mathbf{K}] = 1.$$

The security requirements for an IBKEM we consider here are indistinguishability and anonymity against chosen plaintext and identity attacks (IND-ID-CPA and ANON-ID-CPA). Instead of defining both security notions separately, we define pseudorandom ciphertexts against chosen plaintext and identity attacks (PR-ID-CPA) which means that challenge key and ciphertext are both pseudorandom. Note that PR-ID-CPA trivially implies IND-ID-CPA and ANON-ID-CPA.

We define PR-ID-CPA-security of IBKEM formally via the games given in Figure 7.

<b>Procedure</b> <u>INITIALIZE</u> : $(pk, sk) \leftarrow_{\$} \text{Gen}(1^\lambda)$ Return $pk$  <b>Procedure</b> <u>USKGen</u> ( $id$ ): $\mathcal{Q}_{ID} \leftarrow \mathcal{Q}_{ID} \cup \{id\}$ Return $usk[id] \leftarrow_{\$} \text{USKGen}(sk, id)$	<b>Procedure</b> <u>ENC</u> ( $id^*$ ): //one query $(K^*, C^*) \leftarrow_{\$} \text{Enc}(pk, id^*)$ $K^* \leftarrow_{\$} \mathcal{K}; C^* \leftarrow_{\$} \mathcal{C}$ Return $(K^*, C^*)$  <b>Procedure</b> <u>FINALIZE</u> ( $\beta$ ): Return $(id^* \notin \mathcal{Q}_{ID}) \wedge \beta$
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**Figure 7:** Security Games  $\text{PR-ID-CPA}_{\text{real}}$  and  $\text{PR-ID-CPA}_{\text{rand}}$  for defining PR-ID-CPA-security.

**Definition 4.2 (PR-ID-CPA Security)** An identity-based key encapsulation scheme IBKEM is PR-ID-CPA-secure if for all PPT  $\mathcal{A}$ ,  $\text{Adv}_{\text{IBKEM}}^{\text{pr-id-cpa}}(\mathcal{A}) := |\Pr[\text{PR-ID-CPA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] - \Pr[\text{PR-ID-CPA}_{\text{rand}}^{\mathcal{A}} \Rightarrow 1]|$  is negligible.

## 4.2 The Transformation

Let  $\mathcal{D}_k$  be a matrix distribution that outputs matrices  $\mathbf{A} \in \mathbb{Z}_q^{(k+1) \times k}$ . Let MAC be an affine MAC over  $\mathbb{Z}_q^n$  with message space  $\mathcal{ID}$ . Our IBKEM  $\text{IBKEM}[\text{MAC}, \mathcal{D}_k] = (\text{Gen}, \text{USKGen}, \text{Enc}, \text{Dec})$  for key-space  $\mathcal{K} = \mathbb{G}_T$  and identity space  $\mathcal{ID}$  is defined as follows.

<b>Gen</b> ( $\text{par}$ ): $\mathbf{A} \leftarrow_{\$} \mathcal{D}_k$ $\text{sk}_{\text{MAC}} = (\mathbf{B}, \mathbf{x}_0, \dots, \mathbf{x}_\ell, x'_0, \dots, x'_{\ell'}) \leftarrow_{\$} \text{Gen}_{\text{MAC}}(\text{par})$ For $i = 0, \dots, \ell$ : $\mathbf{Y}_i \leftarrow_{\$} \mathbb{Z}_q^{k \times n}$ ; $\mathbf{Z}_i = (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \cdot \mathbf{A} \in \mathbb{Z}_q^{n \times k}$ For $i = 0, \dots, \ell'$ : $\mathbf{y}'_i \leftarrow_{\$} \mathbb{Z}_q^k$ ; $\mathbf{z}'_i = (\mathbf{y}'_i^\top \mid x'_i) \cdot \mathbf{A} \in \mathbb{Z}_q^{1 \times k}$ $\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, ([\mathbf{z}'_i]_1)_{0 \leq i \leq \ell'})$ $\text{sk} := (\text{sk}_{\text{MAC}}, (\mathbf{Y}_i)_{0 \leq i \leq \ell}, (\mathbf{y}'_i)_{0 \leq i \leq \ell'})$ Return $(\text{pk}, \text{sk})$  <b>USKGen</b> ( $\text{sk}, id$ ): $([t]_2, [u]_2) \leftarrow_{\$} \text{Tag}(\text{sk}_{\text{MAC}}, id)$ $\mathbf{v} = \sum_{i=0}^{\ell} f_i(id) \mathbf{Y}_i \mathbf{t} + \sum_{i=0}^{\ell'} f'_i(id) \mathbf{y}'_i \in \mathbb{Z}_q^k$ Return $usk[id] := ([t]_2, [u]_2, [\mathbf{v}]_2) \in \mathbb{G}_2^{n+1+k}$	<b>Enc</b> ( $\text{pk}, id$ ): $\mathbf{r} \leftarrow_{\$} \mathbb{Z}_q^k$ $\mathbf{c}_0 = \mathbf{A} \mathbf{r} \in \mathbb{Z}_q^{k+1}$ $\mathbf{c}_1 = (\sum_{i=0}^{\ell} f_i(id) \mathbf{Z}_i) \cdot \mathbf{r} \in \mathbb{Z}_q^n$ $K = (\sum_{i=0}^{\ell'} f'_i(id) \mathbf{z}'_i) \cdot \mathbf{r} \in \mathbb{Z}_q$ Return $K = [K]_T$ and $C = ([\mathbf{c}_0]_1, [\mathbf{c}_1]_1) \in \mathbb{G}_1^{n+k+1}$  <b>Dec</b> ( $usk[id], id, C$ ): Parse $usk[id] = ([t]_2, [u]_2, [\mathbf{v}]_2)$ Parse $C = ([\mathbf{c}_0]_1, [\mathbf{c}_1]_1)$ $K = e([\mathbf{c}_0]_1, [\frac{\mathbf{v}}{u}]_2) \cdot e([\mathbf{c}_1]_1, [t]_2)^{-1}$ Return $K \in \mathbb{G}_T$
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The intuition behind our construction is that the values  $[\mathbf{Z}_i]_1, [\mathbf{z}'_i]_1$  from  $\text{pk}$  can be viewed as perfectly hiding commitments to the secrets keys  $\text{sk}_{\text{MAC}} = (\mathbf{x}_1, \dots, \mathbf{x}_\ell, x'_1, \dots, x'_{\ell'})$  of MAC. User secret key generation computes the MAC tag  $\tau = ([t]_2, [u]_2) \leftarrow_{\$} \text{Tag}(\text{sk}_{\text{MAC}})$  plus a “non-interactive zero-knowledge proof”  $[\mathbf{v}]_2$  proving that  $\tau$  was computed correctly with respect to the commitments. As the MAC is affine, the NIZK proof has a very simple structure. The encryption algorithm is derived from a randomized version of the NIZK verification equation. Here we again make use of the affine structure of MAC.

To show correctness of  $\text{IBKEM}[\text{MAC}, \mathcal{D}_k]$ , let  $(K, C)$  be the output of  $\text{Enc}(\text{pk}, id)$  and let  $usk[id]$  be the output of  $\text{USKGen}(\text{sk}, id)$ . By Equation (3) in Section 3, we have

$$e([\mathbf{c}_0]_1, [\frac{\mathbf{v}}{u}]_2) = \left[ (\mathbf{A} \mathbf{r})^\top \cdot \left( \sum_{i=0}^{\ell} f_i(id) \mathbf{Y}_i \mathbf{t} + \sum_{i=0}^{\ell'} f'_i(id) \mathbf{y}'_i \right) \right]_T$$

$$e([\mathbf{c}_1]_1, [t]_2) = \left[ (\mathbf{A} \mathbf{r})^\top \cdot \left( \sum_{i=0}^{\ell} f_i(id) \mathbf{Y}_i \right) \cdot \mathbf{t} \right]_T$$

and the quotient of the two elements yields  $K = (\sum_{i=0}^{\ell'} f'_i(id) \mathbf{z}'_i) \cdot \mathbf{r}$ .

**Theorem 4.3** Under the  $\mathcal{D}_k$ -MDDH assumption relative to  $\text{GGen}$  in  $\mathbb{G}_1$  and the PR-CMA-security of MAC,  $\text{IBKEM}[\text{MAC}, \mathcal{D}_k]$  is a PR-ID-CPA-secure IBKEM. Particularly, for all adversaries  $\mathcal{A}$  there exist

<b>INITIALIZE:</b> <span style="float: right;">// Games <math>G_0</math>-<math>G_4</math></span> $\mathcal{G} \leftarrow_s \text{GGen}(1^\lambda); \mathbf{A} \leftarrow_s \mathcal{D}_k$ $\text{sk}_{\text{MAC}} = (\mathbf{B}, \mathbf{x}_0, \dots, \mathbf{x}_\ell, x'_0, \dots, x'_{\ell'}) \leftarrow_s \text{Gen}_{\text{MAC}}(\mathcal{G})$ For $i = 0, \dots, \ell$ : $\mathbf{Y}_i \leftarrow_s \mathbb{Z}_q^{k \times n}; \mathbf{Z}_i = (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \cdot \mathbf{A} \in \mathbb{Z}_q^{n \times k}$ For $i = 0, \dots, \ell'$ : $\mathbf{y}'_i \leftarrow_s \mathbb{Z}_q^k; \mathbf{z}'_i = (\mathbf{y}'_i{}^\top \mid x'_i) \cdot \mathbf{A} \in \mathbb{Z}_q^{1 \times k}$ $\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, ([\mathbf{z}'_i]_1)_{0 \leq i \leq \ell'})$ $\text{sk} := (\text{sk}_{\text{MAC}}, (\mathbf{Y}_i)_{0 \leq i \leq \ell}, (\mathbf{y}'_i)_{0 \leq i \leq \ell'})$ Return $\text{pk}$  <b>FINALIZE(<math>\beta</math>):</b> <span style="float: right;">// Games <math>G_0</math>-<math>G_4</math></span> Return $(\text{id}^* \notin \mathcal{Q}_{\text{ID}}) \wedge \beta$  <b>USKGEN(<math>\text{id}</math>):</b> <span style="float: right;">// Games <math>G_0</math>-<math>G_2</math>, <math>G_3</math>-<math>G_4</math></span> $\mathcal{Q}_{\text{ID}} = \mathcal{Q}_{\text{ID}} \cup \{\text{id}\}$ $([t]_2, [u]_2) \leftarrow_s \text{Tag}(\text{sk}_{\text{MAC}}, \text{id})$ $\mathbf{v} = \sum_{i=0}^{\ell} f_i(\text{id}) \mathbf{Y}_i \mathbf{t} + \sum_{i=0}^{\ell'} f'_i(\text{id}) \mathbf{y}'_i \in \mathbb{Z}_q^k$ $\mathbf{v}^\top = (\mathbf{t}^\top \sum f_i(\text{id}) \mathbf{Z}_i + \sum f'_i(\text{id}) \mathbf{z}'_i - u \cdot \mathbf{A}) \cdot \bar{\mathbf{A}}^{-1}$ $\text{usk}[\text{id}] := ([t]_2, [u]_2, [\mathbf{v}]_2) \in \mathbb{G}_2^n \times \mathbb{G}_2^1 \times \mathbb{G}_2^k$ Return $\text{usk}[\text{id}]$	<b>ENC(<math>\text{id}^*</math>):</b> <span style="float: right;">// Games <math>G_0</math>, <math>G_1</math>-<math>G_2</math>, <math>G_2</math>, <math>G_3</math></span> $\mathbf{r} \leftarrow_s \mathbb{Z}_q^k;$ $\mathbf{c}_0^* = \mathbf{A} \mathbf{r} \in \mathbb{Z}_q^{k+1}$ $\mathbf{c}_0^* \leftarrow_s \mathbb{Z}_q^{k+1}$ $h \leftarrow_s \mathbb{Z}_q; \mathbf{c}_0^* \leftarrow_s \mathbb{Z}_q^k; \mathbf{c}_0^* := h + \mathbf{A} \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* \in \mathbb{Z}_q$ $\mathbf{c}_1^* = (\sum_{i=0}^{\ell} f_i(\text{id}^*) \mathbf{Z}_i) \mathbf{r} \in \mathbb{Z}_q^n$ $\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \mathbf{c}_0^* \in \mathbb{Z}_q^n$ $\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) (\mathbf{Z}_i \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* + \mathbf{x}_i \cdot h)$ $K^* = \sum_{i=0}^{\ell'} f'_i(\text{id}^*) \mathbf{z}'_i \cdot \mathbf{r} \in \mathbb{Z}_q.$ $K^* = \sum_{i=0}^{\ell'} f'_i(\text{id}^*) (\mathbf{y}'_i{}^\top \mid x'_i) \mathbf{c}_0^* \in \mathbb{Z}_q$ $K^* = \sum_{i=0}^{\ell'} f'_i(\text{id}^*) (\mathbf{z}'_i \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* + x'_i \cdot h)$ Return $K^* = [K^*]_T$ and $C^* = ([c_0^*]_1, [c_1^*]_1)$  <b>ENC(<math>\text{id}^*</math>):</b> <span style="float: right;">// Game <math>G_4</math></span> $K^* \leftarrow_s \mathbb{G}_T$ and $C^* \leftarrow_s \mathbb{G}_1^{n+k+1}$ Return $K^*$ and $C^*$
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**Figure 8:** Games  $G_0$ - $G_4$  for the proof of Theorem 4.3.

adversaries  $\mathcal{B}_1$  and  $\mathcal{B}_2$  with  $\mathbf{T}(\mathcal{B}_1) \approx \mathbf{T}(\mathcal{A}) \approx \mathbf{T}(\mathcal{B}_2)$  and  $\text{Adv}_{\text{IBKEM}[\text{MAC}, \mathcal{D}_k]}^{\text{pr-id-cpa}}(\mathcal{A}) \leq \text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}_1) + \text{Adv}_{\text{MAC}}^{\text{pr-cma}}(\mathcal{B}_2)$ .

**Proof of Theorem 4.3:** We prove Theorem 4.3 by defining a sequence of games  $G_0$ - $G_4$  as in Figure 8. Let  $\mathcal{A}$  be an adversary against the PR-ID-CPA security of  $\text{IBKEM}_{\text{MAC}, \mathcal{D}_k}$ .

**Lemma 4.4**  $\Pr[\text{PR-ID-CPA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] = \Pr[G_1^{\mathcal{A}} \Rightarrow 1] = \Pr[G_0^{\mathcal{A}} \Rightarrow 1]$ .

**Proof:**  $G_0$  is the real attack game. In game  $G_1$ , we change the simulation of  $\mathbf{c}_1^*$  and  $K^*$  in  $\text{ENC}(\text{id}^*)$  by substituting  $\mathbf{Z}_i$  and  $\mathbf{z}'_i$  with their respective definitions:

$$\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) \mathbf{Z}_i \mathbf{r} = \sum_{i=0}^{\ell} f_i(\text{id}^*) (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \mathbf{A} \mathbf{r} = \sum_{i=0}^{\ell} f_i(\text{id}^*) (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \mathbf{c}_0^*$$

and, similarly  $K^* = \sum f'_i(\text{id}^*) (\mathbf{y}'_i{}^\top \mid x'_i) \mathbf{A} \mathbf{r} = \sum f'_i(\text{id}^*) (\mathbf{y}'_i{}^\top \mid x'_i) \mathbf{c}_0^*$ . Thus,  $G_1$  is identical to  $G_0$ .  $\blacksquare$

**Lemma 4.5** *There exists an adversary  $\mathcal{B}_1$  with  $\mathbf{T}(\mathcal{B}_1) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}_1) \geq |\Pr[G_2^{\mathcal{A}} \Rightarrow 1] - \Pr[G_1^{\mathcal{A}} \Rightarrow 1]|$ .*

**Proof:** The only difference between  $G_2$  and  $G_1$  is that  $\mathbf{c}_0^*$  is chosen uniformly at random over  $\mathbb{Z}_q^{k+1}$ . It is easy to see that the joint distribution of  $(\mathcal{G}, [\mathbf{A}]_1, [\mathbf{c}_0^*]_1)$  in  $G_1$  is identical to the real  $\mathcal{D}_k$ -MDDH distribution and  $(\mathcal{G}, [\mathbf{A}]_1, [\mathbf{c}_0^*]_1)$  in  $G_2$  is identical to the random  $\mathcal{D}_k$ -MDDH distribution.

More formally, we build a distinguisher  $\mathcal{B}_1$ .  $\mathcal{B}_1$  takes as input  $(\mathcal{G}, [\mathbf{A}]_1, [\mathbf{b}]_1)$  and it has to distinguish if  $\mathbf{b} = \mathbf{A} \mathbf{w}$  for some random vector  $\mathbf{w} \in \mathbb{Z}_q^k$  or  $\mathbf{b}$  is uniformly random.  $\mathcal{B}_1$  simulates USKGEN and FINALIZE the same way as in  $G_2$  and  $G_1$ . We only describe the simulation of INITIALIZE and ENC in Figure 9. Note that  $\mathcal{B}_1$  knows the secrets  $\mathbf{x}_i, x'_i, \mathbf{Y}_i, \mathbf{y}'_i$  explicitly over  $\mathbb{Z}_q$ . Hence,  $\mathcal{B}_1$  can compute  $([\mathbf{Z}_i]_1, ([\mathbf{z}'_i]_1)_{0 \leq i \leq \ell'})$  from  $\text{pk}$  and  $([\mathbf{c}_1^*]_1, [K^*]_T)$  from the encryption query from  $[\mathbf{A}]_1$  and  $[\mathbf{b}]_1$ . If  $\mathbf{b} = \mathbf{A} \mathbf{w}$  for  $\mathbf{w} \leftarrow_s \mathbb{Z}_q^k$ ,

<b>INITIALIZE:</b> $\text{sk}_{\text{MAC}} = (\mathbf{B}, \mathbf{x}_0, \dots, \mathbf{x}_\ell, x'_0, \dots, x'_{\ell'}) \leftarrow \text{Gen}_{\text{MAC}}(\mathcal{G})$ For $i = 0, \dots, \ell$ : $\mathbf{Y}_i \leftarrow \mathbb{Z}_q^{k \times n}$ ; $\mathbf{Z}_i = (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \cdot \mathbf{A}$ For $i = 0, \dots, \ell'$ : $\mathbf{y}'_i \leftarrow \mathbb{Z}_q^k$ ; $\mathbf{z}'_i = (\mathbf{y}'_i^\top \mid x'_i) \cdot \mathbf{A}$ $\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, ([\mathbf{z}'_i]_1)_{0 \leq i \leq \ell'})$ $\text{sk} := (\text{sk}_{\text{MAC}}, (\mathbf{Y}_i)_{0 \leq i \leq \ell}, (\mathbf{y}'_i)_{0 \leq i \leq \ell'})$ Return pk	<b>ENC(id*):</b> $\mathbf{c}_0^* := \mathbf{b} \in \mathbb{Z}_q^{k+1}$ $\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*)(\mathbf{Y}_i^\top \mid \mathbf{x}_i) \mathbf{c}_0^* \in \mathbb{Z}_q^n$ $K^* = \sum_{i=0}^{\ell'} f'_i(\text{id}^*)(\mathbf{y}'_i^\top \mid x'_i) \mathbf{c}_0^* \in \mathbb{Z}_q$ Return $\mathbf{K}^* = [K^*]_T$ and $\mathbf{C}^* = ([\mathbf{c}_0^*]_1, [\mathbf{c}_1^*]_1)$
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**Figure 9:** Description of  $\mathcal{B}_1(\mathcal{G}, [\mathbf{A}]_1, [\mathbf{b}]_1)$  for the proof of Lemma 4.5.

then the simulation is distributed as in  $\mathcal{G}_1$ . If  $\mathbf{b}$  is uniformly random, then the simulation is distributed as in  $\mathcal{G}_2$ . ■

Following the intuition of the construction, in Game  $\mathcal{G}_3$ , we simulate the values  $\mathbf{v}$  computed in the USKGEN algorithm using a “perfect zero-knowledge” simulator, and ENC is simulated without using  $(\mathbf{Y}_i)_{0 \leq i \leq \ell}$  and  $(\mathbf{y}'_i)_{0 \leq i \leq \ell'}$ , which is ready to conclude the proof by using the PR-CMA security of MAC.

**Lemma 4.6**  $\Pr[\mathcal{G}_3^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathcal{G}_2^{\mathcal{A}} \Rightarrow 1]$ .

**Proof:**  $\mathcal{G}_3$  does not use  $(\mathbf{Y}_i)_{0 \leq i \leq \ell}$  and  $(\mathbf{y}'_i)_{0 \leq i \leq \ell'}$  any more. We now show that the changes are purely conceptual. By  $\mathbf{Z}_i = (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \mathbf{A}$ , we have  $\mathbf{Y}_i^\top = (\mathbf{Z}_i - \mathbf{x}_i \cdot \mathbf{A}) \cdot (\bar{\mathbf{A}})^{-1}$ , and similarly we have  $\mathbf{y}'_i^\top = (\mathbf{z}'_i - x'_i \cdot \mathbf{A}) \cdot (\bar{\mathbf{A}})^{-1}$ . For USKGEN(id), by substituting  $\mathbf{Y}_i^\top$  and  $\mathbf{y}'_i^\top$ , we obtain

$$\begin{aligned}
\mathbf{v}^\top &= \left( \mathbf{t}^\top \sum f_i(\text{id})(\mathbf{Z}_i - \mathbf{x}_i \cdot \mathbf{A}) + \sum f'_i(\text{id})(\mathbf{z}'_i - x'_i \cdot \mathbf{A}) \right) (\bar{\mathbf{A}})^{-1} \\
&= \left( \mathbf{t}^\top \sum f_i(\text{id}) \mathbf{Z}_i + \sum f'_i(\text{id}) \mathbf{z}'_i - \underbrace{\left( \mathbf{t}^\top \sum f_i(\text{id}) \mathbf{x}_i + \sum f'_i(\text{id}) x'_i \right) \cdot \mathbf{A}}_u \right) (\bar{\mathbf{A}})^{-1}.
\end{aligned}$$

Note that we can compute  $[\mathbf{v}]_2$  in  $\mathcal{G}_2$ , since  $\mathbf{A}$ ,  $\mathbf{z}'_i$  and  $\mathbf{Z}_i$  are known explicitly over  $\mathbb{Z}_q$  and  $[\mathbf{t}]_2$  and  $[u]_2$  are known.

As for the distribution of ENC(id\*), it is easy to see that  $\mathbf{c}_0^*$  is uniformly random, as in  $\mathcal{G}_2$ . By  $h = \underline{\mathbf{c}}_0^* - \mathbf{A} \cdot \bar{\mathbf{A}}^{-1} \bar{\mathbf{c}}_0^*$ , we have

$$\begin{aligned}
\mathbf{c}_1^* &= \sum f_i(\text{id}_i^*)(\mathbf{Z}_i \cdot \bar{\mathbf{A}}^{-1} \bar{\mathbf{c}}_0^* + \mathbf{x}_i \cdot (\underline{\mathbf{c}}_0^* - \mathbf{A} \cdot \bar{\mathbf{A}}^{-1} \bar{\mathbf{c}}_0^*)) \\
&= \sum f_i(\text{id}_i^*)(\mathbf{Y}_i^\top \bar{\mathbf{A}} + \mathbf{x}_i \mathbf{A}) \cdot \bar{\mathbf{A}}^{-1} \bar{\mathbf{c}}_0^* + \mathbf{x}_i \cdot (\underline{\mathbf{c}}_0^* - \mathbf{A} \cdot \bar{\mathbf{A}}^{-1} \bar{\mathbf{c}}_0^*) \\
&= \sum f_i(\text{id}_i^*)(\mathbf{Y}_i^\top \mid \mathbf{x}_i) \mathbf{c}_0^*
\end{aligned}$$

and  $\mathbf{c}_1^*$  is distributed as in  $\mathcal{G}_2$ . The distribution of  $\mathbf{K}^*$  can be analyzed with a similar argument. ■

**Lemma 4.7** *There exists an adversary  $\mathcal{B}_2$  with  $\mathbf{T}(\mathcal{B}_2) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\text{MAC}}^{\text{pr-cma}}(\mathcal{B}_2) \geq |\Pr[\mathcal{G}_4^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathcal{G}_3^{\mathcal{A}} \Rightarrow 1]|$ .*

**Proof:** In  $\mathcal{G}_4$ , we answer the ENC(id\*) query by choosing random  $\mathbf{K}^*$  and  $\mathbf{C}^*$ . We construct an adversary  $\mathcal{B}_2$  in Figure 10 to show the differences between  $\mathcal{G}_4$  and  $\mathcal{G}_3$  can be bounded by the advantage of breaking PR-CMA security of MAC. Intuitively, the reduction to PR-CMA security of the symmetric primitive MAC



<p><u>INITIALIZE:</u>  <math>\mathbf{A} \leftarrow_{\\$} \mathcal{D}_k</math>  <math>\varepsilon \leftarrow_{\\$} \text{INITIALIZE}_{\text{MAC}}</math>  For <math>i = 0, \dots, \ell</math>: <math>\mathbf{Z}_i \leftarrow_{\\$} \mathbb{Z}_q^{n \times k}</math>  For <math>i = 0, \dots, \ell'</math>: <math>\mathbf{z}'_i \leftarrow_{\\$} \mathbb{Z}_q^{1 \times k}</math>  <math>\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, ([\mathbf{z}'_i]_1)_{0 \leq i \leq \ell'})</math>  Return <math>\text{pk}</math></p> <p><u>USKGEN(id):</u>  <math>\mathcal{Q}_{\text{ID}} = \mathcal{Q}_{\text{ID}} \cup \{\text{id}\}</math>  <math>([\mathbf{t}]_2, [u]_2) \leftarrow_{\\$} \text{EVAL}(\text{id})</math>  <math>\mathbf{v}^\top = (\mathbf{t}^\top \sum f_i(\text{id}) \mathbf{Z}_i + \sum f'_i(\text{id}) \mathbf{z}'_i - u \cdot \mathbf{A}) \cdot (\bar{\mathbf{A}})^{-1}</math>  <math>\text{usk}[\text{id}] := ([\mathbf{t}]_2, [u]_2, [\mathbf{v}]_2) \in \mathbb{G}_2^n \times \mathbb{G}_2^1 \times \mathbb{G}_2^k</math>  Return <math>\text{usk}[\text{id}]</math></p>	<p><u>ENC(id*):</u> //one query  <math>([h]_1, [\mathbf{h}_0]_1, [h_1]_T) \leftarrow_{\\$} \text{CHAL}(\text{id}^*)</math>  <math>\mathbf{c}_0^* \leftarrow_{\\$} \mathbb{Z}_q^k</math>  <math>\mathbf{c}_0^* = h + \mathbf{A} \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* \in \mathbb{Z}_q</math>  <math>\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) \mathbf{Z}_i \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* + \mathbf{h}_0</math>  <math>K^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) \mathbf{z}'_i \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* + h_1</math>  Return <math>\mathbf{K}^* = [K^*]_T</math> and <math>\mathbf{C}^* = ([\mathbf{c}_0^*]_1, [\mathbf{c}_1^*]_1)</math></p> <p><u>FINALIZE(<math>\beta'</math>):</u>  Return <math>(\text{id}^* \notin \mathcal{Q}_{\text{ID}}) \wedge \text{FINALIZE}_{\text{MAC}}(\beta')</math></p>
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**Figure 10:** Description of  $\mathcal{B}_2$  (having access to the oracles  $\text{INITIALIZE}_{\text{MAC}}, \text{EVAL}, \text{CHAL}, \text{FINALIZE}_{\text{MAC}}$  of the  $\text{PR-CMA}_{\text{real}}/\text{PR-CMA}_{\text{rand}}$  games of Figure 1) for the proof of Lemma 4.7.

can be carried out as in both  $\mathcal{G}_3$  and  $\mathcal{G}_4$ ,  $\text{sk}_{\text{MAC}}$  (i.e.,  $\mathbf{x}_i$  and  $x'_i$ ) is perfectly hidden until  $\mathcal{B}_2$ 's call to  $\text{ENC}(\text{id}^*)$ .

If  $(\mathbf{h}_0, h_1)$  is uniform (i.e.,  $\mathcal{B}_2$  is in Game  $\text{PR-CMA}_{\text{rand}}$ ) then the view of  $\mathcal{A}$  is the same as in  $\mathcal{G}_4$ . If  $(\mathbf{h}_0, h_1)$  is real (i.e.,  $\mathcal{B}_2$  is in Game  $\text{PR-CMA}_{\text{real}}$ ) then the view of  $\mathcal{A}$  is the same as in  $\mathcal{G}_3$ . ■

The proof of Theorem 4.3 follows by Lemmas 4.4-4.7 and observing that  $\mathcal{G}_4 = \text{PR-ID-CPA}_{\text{rand}}$ . ■

## 5 Hierarchical Identity-based Encryption from Delegatable Affine MACs

In this section, we will define syntax and security requirements of *delegatable* affine MACs and describe our transformation  $\text{HIBE}[\text{MAC}, \mathcal{D}_k]$  from delegatable affine MACs to HIBE based on any  $\mathcal{D}_k$ -MDDH assumption.

### 5.1 Delegatable Affine MACs

**Definition 5.1** An affine MAC over  $\mathbb{Z}_q^n$  (Definition 3.1) is *delegatable*, if the message space is  $\mathcal{M} = \mathcal{B}^{\leq m}$  for some finite base set  $\mathcal{B}$ ,  $\ell' = 0$  with  $f'_0(\mathbf{m}) = 1$ , and there exists a public function  $l : \mathcal{M} \rightarrow \{0, \dots, \ell\}$  such that for all  $\mathbf{m}' \in \mathcal{M}$  with  $\mathbf{m}' = (\mathbf{m}_1, \dots, \mathbf{m}_{p+1}) \in \mathcal{B}^{p+1}$  and length  $p$  prefix  $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_p)$  of  $\mathbf{m}'$ , we have  $l(\mathbf{m}) \leq l(\mathbf{m}')$  and

$$f_i(\mathbf{m}') = \begin{cases} f_i(\mathbf{m}) & 0 \leq i \leq l(\mathbf{m}) \\ 0 & l(\mathbf{m}') < i \leq \ell \end{cases}.$$

Note that for a delegatable MAC, equation (3) simplifies to

$$u = \left( \sum_{i=0}^{l(\mathbf{m})} f_i(\mathbf{m}) \mathbf{x}_i^\top + \sum_{i=l(\mathbf{m})+1}^{l(\mathbf{m}')} f_i(\mathbf{m}') \mathbf{x}_i^\top \right) \mathbf{t} + f'_0(\mathbf{m}) x'_0.$$

Intuitively, this property will be used for HIBE user secret key delegation.

**SECURITY REQUIREMENTS.** Let  $\text{MAC}$  be a delegatable affine MAC over  $\mathbb{Z}_q^n$  with message space  $\mathcal{M} = \mathcal{B}^{\leq m} := \bigcup_{i=1}^m \mathcal{B}^i$ . To build a HIBE, we require a new notion denoted as  $\text{HPR}_0\text{-CMA}$  security. It differs from  $\text{PR-CMA}$  security in two ways. Firstly, additional values needed for HIBE delegation are provided

<b>INITIALIZE:</b> $\text{sk}_{\text{MAC}} = (\mathbf{B}, (\mathbf{x}_i)_{0 \leq i \leq \ell}, x'_0) \leftarrow_{\$} \text{Gen}_{\text{MAC}}(\text{par})$ Return $([\mathbf{B}]_2, ([\mathbf{x}_i^\top \mathbf{B}]_2)_{0 \leq i \leq \ell})$  <b>EVAL(m):</b> $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ $([\mathbf{t}]_2, [u]_2) \leftarrow_{\$} \text{Tag}(\text{sk}_{\text{MAC}}, \mathbf{m})$ For $i = (\mathbf{m}) + 1, \dots, \ell$ : $d_i = \mathbf{x}_i^\top \mathbf{t} \in \mathbb{Z}_q$ ; $d'_i = \mathbf{x}_i^\top \mathbf{t}' \in \mathbb{Z}_q$ Return $([\mathbf{t}]_2, [u]_2, [\mathbf{t}']_2, [u']_2, ([d_i]_2)_{(\mathbf{m})+1 \leq i \leq \ell})$	<b>CHAL(m*):</b> // one query $h \leftarrow_{\$} \mathbb{Z}_q$ $\mathbf{h}_0 = \sum f_i(\mathbf{m}_i^*) \mathbf{x}_i \cdot h \in \mathbb{Z}_q^n$ $h_1 = x'_0 \cdot h \in \mathbb{Z}_q$ $[h_1] \leftarrow_{\$} \mathbb{Z}_q$ Return $([h]_1, [\mathbf{h}_0]_1, [h_1]_T)$  <b>FINALIZE(<math>\beta \in \{0, 1\}</math>):</b> Return $\beta \wedge (\text{Prefix}(\mathbf{m}^*) \cap \mathcal{Q}_{\mathcal{M}} = \emptyset)$
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**Figure 11:** Games  $\text{HPR-CMA}_{\text{real}}$ , and  $\text{HPR}_0\text{-CMA}_{\text{rand}}$  for defining  $\text{HPR}_0\text{-CMA}$  security.

to the adversary through the call to INITIALIZE and EVAL. Secondly, CHAL always returns a real  $\mathbf{h}_0$  which is the reason why our HIBE is not anonymous. (In fact, the additional values actually allow the adversary to distinguish real from random  $\mathbf{h}_0$ .)

Let  $\mathcal{G} = (\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, q, g_1, g_2, e)$  be an asymmetric pairing group such that  $(\mathbb{G}_2, g_2, q)$  is contained in  $\text{par}$ . Consider the games from Figure 11.

**Definition 5.2** A delegatable affine MAC over  $\mathbb{Z}_q^n$  is  $\text{HPR}_0\text{-CMA}$ -secure if for all PPT  $\mathcal{A}$ ,  $\text{Adv}_{\text{MAC}}^{\text{hpr}_0\text{-cma}}(\mathcal{A}) := \Pr[\text{HPR-CMA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] - \Pr[\text{HPR}_0\text{-CMA}_{\text{rand}}^{\mathcal{A}} \Rightarrow 1]$  is negligible.

## 5.2 Examples of Delegatable Affine MACs

We first note that  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$  from Section 3 with message space  $\mathcal{M} = \{0, 1\}^{\leq m}$  is delegatable.

**Theorem 5.3** Under the  $\mathcal{D}_k\text{-MDDH}$  assumption,  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$  is tightly  $\text{HPR}_0\text{-CMA}$  secure. In particular, for all adversaries  $\mathcal{A}$  there exists an adversary  $\mathcal{D}$  with  $\mathbf{T}(\mathcal{A}) \approx \mathbf{T}(\mathcal{D})$  and  $\text{Adv}_{\text{MAC}_{\text{NR}}[\mathcal{D}_k]}^{\text{hpr}_0\text{-cma}}(\mathcal{A}) \leq 6m(\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{D}) - 1/(q-1))$ .

The proof is similar to that of Theorem 3.3, with the difference that the reduction between games  $\mathbf{G}_i$  and  $\mathbf{G}_{i-1}$  now has to guess  $\mathbf{m}_i^* \in \{0, 1, \perp\}$ , where  $\perp$  means that  $|\mathbf{m}^*| < i$ . Furthermore,  $\mathbf{h}_0$  from  $\text{CHAL}(\mathbf{m}^*)$  is not pseudorandom in the delegatable case, since  $([\mathbf{B}]_2, ([\mathbf{x}_i^\top \mathbf{B}]_2)_{0 \leq i \leq m})$  are disclosed from INITIALIZE and then it is easy to check if  $\mathbf{h}_0$  is well-formed under  $\mathbf{m}^*$  by using the pairing. A formal proof of Theorem 5.3 is postponed to Appendix A.1.

We now turn to  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  from Section 3 with message space  $\mathcal{M} = \mathcal{B}^{\leq m} = (\mathbb{Z}_q^*)^{\leq m}$ . Again, it can be verified to be delegatable. One should remark the change on  $\mathcal{B}$ , where we now define  $\mathcal{B} = \mathbb{Z}_q^*$  to avoid having a collision between the MAC of  $\mathbf{m}$  and the MAC of  $\mathbf{m}||0$ .

**Theorem 5.4** Under the  $\mathcal{D}_k\text{-MDDH}$  assumption,  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  is  $\text{HPR}_0\text{-CMA}$ -secure. In particular, for all adversaries  $\mathcal{A}$  there exists an adversary  $\mathcal{D}$  with  $\mathbf{T}(\mathcal{A}) \approx \mathbf{T}(\mathcal{D})$  and  $\text{Adv}_{\text{MAC}_{\text{HPS}}[\mathcal{D}_k]}^{\text{hpr}_0\text{-cma}}(\mathcal{A}) \leq 2Q(\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{D}) + 1/q)$ , where  $Q$  is the maximal number of queries to EVAL( $\cdot$ ).

The proof is postponed to Appendix A.2

## 5.3 Hierarchical Identity-Based Key Encapsulation

We recall syntax and security of a hierarchical identity-based key encapsulation mechanism (HIBKEM).

**Definition 5.5 (Hierarchical Identity-Based Key Encapsulation Mechanism)** A hierarchical identity-based key encapsulation mechanism (HIBE) HIBKEM consists of three PPT algorithms  $\text{HIBKEM} = (\text{Gen}, \text{USKDel}, \text{USKGen}, \text{Enc}, \text{Dec})$  with the following properties.

- The probabilistic key generation algorithm  $\text{Gen}(1^\lambda)$  returns the (master) public/secret key and delegation key  $(\text{pk}, \text{sk}, \text{dk})$ . Note that for some of our constructions  $\text{dk}$  is empty. We assume that  $\text{pk}$  implicitly defines a message space  $\mathcal{M}$  and hierarchical identity space  $\mathcal{ID} = \mathcal{B}^{\leq m}$ , for some base identity set  $\mathcal{B}$ .

<b>INITIALIZE:</b> $(pk, sk, dk) \leftarrow_s \text{Gen}(1^\lambda)$ Return $(pk, dk)$  <b>USKGen(id):</b> $\mathcal{Q}_{ID} \leftarrow \mathcal{Q}_{ID} \cup \{id\}$ Return $(usk[id], udk[id]) \leftarrow_s \text{USKGen}(sk, id)$	<b>ENC(id*):</b> //one query $(K^*, C^*) \leftarrow_s \text{Enc}(pk, id^*)$ $K^* \leftarrow_s \mathcal{K}$ $(K^*, C^*) \leftarrow_s \mathcal{K} \times \mathcal{C}$ Return $(K^*, C^*)$  <b>FINALIZE(<math>\beta \in \{0, 1\}</math>):</b> Return $(\text{Prefix}(id^*) \cap \mathcal{Q}_{ID} = \emptyset) \wedge \beta$
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**Figure 12:** Games  $\text{PR-HID-CPA}_{\text{real}}$ ,  $\text{IND-HID-CPA}_{\text{rand}}$ , and  $\text{PR-HID-CPA}_{\text{rand}}$  for defining IND-HID-CPA and PR-HID-CPA-security. For any identity  $id \in \mathcal{B}^p$ ,  $\text{Prefix}(id)$  denotes the set of all prefixes of  $id$  (where  $|\text{Prefix}(id)| = O(|\mathcal{B}|^p)$ ).

- The probabilistic user secret key generation algorithm  $\text{USKGen}(sk, id)$  returns a secret key  $usk[id]$  and a delegation value  $udk[id]$  for hierarchical identity  $id \in \mathcal{ID}$ .
- The probabilistic key delegation algorithm  $\text{USKDel}(dk, usk[id], udk[id], id \in \mathcal{B}^p, id_{p+1} \in \mathcal{B})$  returns a user secret key  $usk[id|id_{p+1}]$  for the hierarchical identity  $id' = id \mid id_{p+1} \in \mathcal{B}^{p+1}$  and the user delegation key  $udk[id']$ . We require  $1 \leq |id| \leq m - 1$ .
- The probabilistic encapsulation algorithm  $\text{Enc}(pk, id)$  returns a symmetric key  $K \in \mathcal{K}$  together with a ciphertext  $C$  with respect to the hierarchical identity  $id \in \mathcal{ID}$ .
- The deterministic decapsulation algorithm  $\text{Dec}(usk[id], id, C)$  returns a decapsulated key  $K \in \mathcal{K}$  or the reject symbol  $\perp$ .

For correctness we require that for all  $\lambda \in \mathbb{N}$ , all pairs  $(pk, sk)$  generated by  $\text{Gen}(1^\lambda)$ , all  $id \in \mathcal{ID}$ , all  $usk[id]$  generated by  $\text{USKGen}(sk, id)$  and all  $(K, c)$  generated by  $\text{Enc}(pk, id)$ :

$$\Pr[\text{Dec}(usk[id], id, C) = K] = 1.$$

Moreover, we also require the distribution of  $usk[id|id_{p+1}]$  from  $\text{USKDel}(usk[id], udk[id], id, id_{p+1})$  is identical to the one from  $\text{USKGen}(sk, id|id_{p+1})$ .

In our HIBKEM definition we make the delegation key  $dk$  and the user delegation key  $udk[id]$  explicit to make our constructions more readable. We define indistinguishability (IND-HID-CPA) and pseudorandom ciphertexts (PR-HID-CPA) against adaptively chosen identity and plaintext attacks for a HIBKEM via games  $\text{PR-HID-CPA}_{\text{real}}$ ,  $\text{IND-HID-CPA}_{\text{rand}}$  and  $\text{PR-HID-CPA}_{\text{rand}}$  from Figure 12.

**Definition 5.6 (IND-HID-CPA and PR-HID-CPA Security)** A hierarchical identity-based key encapsulation scheme HIBKEM is IND-HID-CPA-secure if for all PPT  $\mathcal{A}$ ,  $\text{Adv}_{\text{HIBKEM}}^{\text{ind-hid-cpa}}(\mathcal{A}) := |\Pr[\text{PR-HID-CPA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] - \Pr[\text{IND-HID-CPA}_{\text{rand}}^{\mathcal{A}}]|$  is negligible. It is PR-HID-CPA-secure if for all PPT  $\mathcal{A}$ ,  $\text{Adv}_{\text{HIBKEM}}^{\text{pr-hid-cpa}}(\mathcal{A}) := |\Pr[\text{PR-HID-CPA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] - \Pr[\text{PR-HID-CPA}_{\text{rand}}^{\mathcal{A}}]|$  is negligible.

Note that PR-HID-CPA trivially implies IND-HID-CPA and anonymity of HIBKEM.

## 5.4 The Transformation

Let  $\mathcal{D}_k$  be a matrix distribution that outputs matrices  $\mathbf{A} \in \mathbb{Z}_q^{(k+1) \times k}$ . Let MAC be a delegatable affine MAC over  $\mathbb{Z}_q^n$  with message space  $\mathcal{M} = \mathcal{B}^{\leq m}$ . Our HIBKEM[MAC,  $\mathcal{D}_k$ ] = (Gen, USKGen, USKDel, Enc, Dec) for key-space  $\mathcal{K} = \mathbb{G}_T$  and hierarchical identity space  $\mathcal{ID} = \mathcal{M} = \mathcal{B}^{\leq m}$  is defined as in Fig. 13. Compared to the IBE construction from Sect. 4, the main difference is that Gen also returns a delegation key  $dk$  which allows re-randomization of every  $usk[id]$ . Further, USKGen also outputs user delegation keys  $udk[id]$  allowing USKDel to delegate.

<p><b>Gen(par):</b>  <math>\mathbf{A} \leftarrow_{\\$} \mathcal{D}_k</math>  <math>\text{sk}_{\text{MAC}} = (\mathbf{B}, \mathbf{x}_0, \dots, \mathbf{x}_\ell, x'_0) \leftarrow_{\\$} \text{Gen}_{\text{MAC}}(\text{par})</math>  For <math>i = 0, \dots, \ell</math>:  <math>\mathbf{Y}_i \leftarrow_{\\$} \mathbb{Z}_q^{k \times n}; \mathbf{Z}_i = (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \cdot \mathbf{A} \in \mathbb{Z}_q^{n \times k}</math>  <math>\mathbf{d}_i = \mathbf{x}_i^\top \cdot \mathbf{B} \in \mathbb{Z}_q^{n'}; \mathbf{E}_i = \mathbf{Y}_i \cdot \mathbf{B} \in \mathbb{Z}_q^{k \times n'}</math>  <math>\mathbf{y}'_0 \leftarrow_{\\$} \mathbb{Z}_q^k; \mathbf{z}'_0 = (\mathbf{y}'_0^\top \mid x'_0) \cdot \mathbf{A} \in \mathbb{Z}_q^{1 \times k}</math>  <math>\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, [\mathbf{z}'_0]_1)</math>  <math>\text{dk} := ([\mathbf{B}]_2, ([\mathbf{d}_i]_2, [\mathbf{E}_i]_2)_{0 \leq i \leq \ell})</math>  <math>\text{sk} := (\text{sk}_{\text{MAC}}, (\mathbf{Y}_i)_{0 \leq i \leq \ell}, \mathbf{y}'_0)</math>  Return <math>(\text{pk}, \text{dk}, \text{sk})</math></p> <p><b>USKGen(sk, id <math>\in \mathcal{ID}</math>):</b>  <math>([\mathbf{t}]_2, [\mathbf{u}]_2) \leftarrow_{\\$} \text{Tag}(\text{sk}_{\text{MAC}}, \text{id})</math>  // <math>\mathbf{t} \in \mathbb{Z}_q^n; u = \sum f_i(\text{id}) \mathbf{x}_i^\top \mathbf{t} + x'_0 \in \mathbb{Z}_q</math>  <math>\mathbf{v} = \sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Y}_i \mathbf{t} + \mathbf{y}'_0 \in \mathbb{Z}_q^k</math>  For <math>i = l(\text{id}) + 1, \dots, \ell</math>:  <math>\mathbf{d}_i = \mathbf{x}_i^\top \mathbf{t} \in \mathbb{Z}_q</math>  <math>\mathbf{e}_i = \mathbf{Y}_i \mathbf{t} \in \mathbb{Z}_q^k</math>  <math>\text{usk}[\text{id}] := ([\mathbf{t}]_2, [\mathbf{u}]_2, [\mathbf{v}]_2) \in \mathbb{G}_2^n \times \mathbb{G}_2^1 \times \mathbb{G}_2^k</math>  <math>\text{udk}[\text{id}] := ([\mathbf{d}_i]_2, [\mathbf{e}_i]_2)_{l(\text{id}) &lt; i \leq \ell} \in (\mathbb{G}_2^{1+k})^{\ell - l(\text{id})}</math>  Return <math>(\text{usk}[\text{id}], \text{udk}[\text{id}])</math></p> <p><b>Enc(pk, id):</b>  <math>\mathbf{r} \leftarrow_{\\$} \mathbb{Z}_q^k</math>  <math>\mathbf{c}_0 = \mathbf{A} \mathbf{r} \in \mathbb{Z}_q^{k+1}</math>  <math>\mathbf{c}_1 = (\sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Z}_i) \cdot \mathbf{r} \in \mathbb{Z}_q^n</math>  <math>K = \mathbf{z}'_0 \cdot \mathbf{r} \in \mathbb{Z}_q</math>  Return <math>\mathbf{K} = [K]_T</math> and <math>\mathbf{C} = ([\mathbf{c}_0]_1, [\mathbf{c}_1]_1)</math></p>	<p><b>USKDel(usk[id], udk[id], id <math>\in \mathcal{B}^p</math>, id<sub>p+1</sub> <math>\in \mathcal{B}</math>):</b>  If <math>p \geq m</math>, then return <math>\perp</math>  <math>\text{id}' := (\text{id}_1, \dots, \text{id}_p, \text{id}_{p+1}) \in \mathcal{B}^{p+1}</math>  //Delegation of <math>u</math> and <math>\mathbf{v}</math>:  <math>\hat{u} = u + \sum_{i=l(\text{id})+1}^{l(\text{id}')} f_i(\text{id}') d_i \in \mathbb{Z}_q</math>  <math>\hat{\mathbf{v}} = \mathbf{v} + \sum_{i=l(\text{id})+1}^{l(\text{id}')} f_i(\text{id}') \mathbf{e}_i \in \mathbb{Z}_q^k</math>  //Rerandomization of <math>\hat{u}</math> and <math>\hat{\mathbf{v}}</math>:  <math>\mathbf{s}' \leftarrow_{\\$} \mathbb{Z}_q^{n'}</math>  <math>\mathbf{t}' = \mathbf{t} + \mathbf{B} \mathbf{s}' \in \mathbb{Z}_q^n</math>  <math>u' = \hat{u} + \sum_{i=0}^{l(\text{id}')} f_i(\text{id}') \mathbf{d}_i \mathbf{s}' \in \mathbb{Z}_q</math>  <math>\mathbf{v}' = \hat{\mathbf{v}} + \sum_{i=0}^{l(\text{id}')} f_i(\text{id}') \mathbf{E}_i \mathbf{s}' \in \mathbb{Z}_q^k</math>  //Rerandomization of <math>d'_i</math> and <math>\mathbf{e}'_i</math>:  For <math>i = l(\text{id}') + 1, \dots, \ell</math>:  <math>d'_i = d_i + \mathbf{d}_i \mathbf{s}' \in \mathbb{Z}_q</math>  <math>\mathbf{e}'_i = \mathbf{e}_i + \mathbf{E}_i \mathbf{s}' \in \mathbb{Z}_q^k</math>  <math>\text{usk}[\text{id}'] := ([\mathbf{t}']_2, [\mathbf{u}']_2, [\mathbf{v}']_2) \in \mathbb{G}_2^n \times \mathbb{G}_2^1 \times \mathbb{G}_2^k</math>  <math>\text{udk}[\text{id}'] := ([d'_i]_2, [\mathbf{e}'_i]_2)_{l(\text{id}') &lt; i \leq \ell} \in (\mathbb{G}_2^{1+k})^{\ell - l(\text{id}')}</math>  Return <math>(\text{usk}[\text{id}'], \text{udk}[\text{id}'])</math></p> <p><b>Dec(usk[id], id, C):</b>  Parse <math>\text{usk}[\text{id}] = ([\mathbf{t}]_2, [\mathbf{u}]_2, [\mathbf{v}]_2)</math>  Parse <math>\mathbf{C} = ([\mathbf{c}_0]_1, [\mathbf{c}_1]_1)</math>  <math>\mathbf{K} = e([\mathbf{c}_0]_1, \begin{bmatrix} \mathbf{v} \\ u \end{bmatrix}_2) \cdot e([\mathbf{c}_1]_1, [\mathbf{t}]_2)^{-1}</math>  Return <math>\mathbf{K} \in \mathbb{G}_T</math></p>
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**Figure 13:** Definition of the transformation  $\text{HIBKEM}[\text{MAC}, \mathcal{D}_k]$ .

To show correctness of  $\text{HIBKEM}[\text{MAC}, \mathcal{D}_k]$ , first note that  $(\hat{u}, \hat{\mathbf{v}})$  computed in  $\text{USKDel}$  is a correct user secret key for  $\text{id}'$ ,  $\hat{u} = \sum_{i=0}^{l(\text{id}')} f_i(\text{id}') \mathbf{x}_i^\top \mathbf{t} + x'_0$  and  $\hat{\mathbf{v}} = \sum_{i=0}^{l(\text{id}')} f_i(\text{id}') \mathbf{Y}_i \mathbf{t} + \mathbf{y}'_0$ . In the next step they get rerandomized as  $u' = \sum_{i=0}^{l(\text{id}')} f_i(\text{id}') \mathbf{x}_i^\top (\mathbf{t} + \mathbf{B} \mathbf{s}')$  and  $\mathbf{v}' = \sum_{i=0}^{l(\text{id}')} f_i(\text{id}') \mathbf{Y}_i (\mathbf{t} + \mathbf{B} \mathbf{s}') + \mathbf{y}'_0$ . Consequently,  $\text{usk}[\text{id}']$  from  $\text{USKDel}$  has the same distribution as the one output by  $\text{USKGen}$ . By applying the similar correctness argument from  $\text{HIBKEM}[\text{MAC}, \mathcal{D}_k]$ , we can show that a correctly generated ciphertext can be correctly decapsulated by using a correct user secret key.

The next theorem shows IND-HID-CPA-security of our construction. Its proof is postponed to Appendix B.1. We remark that  $\text{HIBKEM}[\text{MAC}, \mathcal{D}_k]$  can never be anonymous as one can always check whether  $\mathbf{c}_0 \cdot \sum f_i(\text{id}) (\mathbf{E}_i^\top \parallel \mathbf{d}_i) = c_1 \cdot \mathbf{B}$  using the pairing.

**Theorem 5.7** *If MAC is HPR<sub>0</sub>-CMA-secure and the  $\mathcal{D}_k$ -MDDH assumption holds in  $\mathbb{G}_1$  then  $\text{HIBKEM}[\text{MAC}, \mathcal{D}_k]$  is IND-HID-CPA secure. For all adversaries  $\mathcal{A}$  there exist adversaries  $\mathcal{B}_1$  and  $\mathcal{B}_2$  with  $\mathbf{T}(\mathcal{B}_1) \approx \mathbf{T}(\mathcal{A}) \approx \mathbf{T}(\mathcal{B}_2)$  and  $\text{Adv}_{\text{HIBKEM}[\text{MAC}, \mathcal{D}_k]}^{\text{ind-hid-cpa}}(\mathcal{A}) \leq \text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}_1) + \text{Adv}_{\text{MAC}}^{\text{hpr}_0\text{-cma}}(\mathcal{B}_2)$ .*

## 5.5 Anonymity-preserving Transformation

In this section, we give an alternative (but less efficient) transformation, which is anonymity-preserving. Our transformation is based on the notion of APR-CMA-security (anonymity-preserving pseudorandomness against chosen-message attacks) for a delegatable affine MAC  $\text{MAC}$  over  $\mathbb{Z}_q^n$  with message space  $\mathcal{M} = \mathcal{B}^{\leq m} := \bigcup_{i=1}^m \mathcal{B}^i$ . It differs from HPR-CMA-security (Section 5.1) in the sense that  $\text{EVAL}(\mathbf{m})$  will output the terms required for  $\text{usk}$  rerandomization, not  $\text{INITIALIZE}$ . Let  $\mathcal{G} = (\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, q, g_1, g_2, e)$  be

<p><b>INITIALIZE:</b>  <math>\text{sk}_{\text{MAC}} = (\mathbf{B}, (\mathbf{x}_i)_{0 \leq i \leq \ell}, \mathbf{x}'_0) \leftarrow_{\\$} \text{Gen}_{\text{MAC}}(\text{par})</math>  Return <math>\varepsilon</math></p> <p><b>EVAL(m):</b>  <math>\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{m\}</math>  <math>([t]_2, [u]_2) \leftarrow_{\\$} \text{Tag}(\text{sk}_{\text{MAC}}, m)</math>  <math>\mathbf{S} \leftarrow_{\\$} \mathbb{Z}_q^{n' \times \mu}; \mathbf{T} = \mathbf{B} \cdot \mathbf{S} \in \mathbb{Z}_q^{n \times \mu}</math>  <math>\mathbf{u} = \sum_{i=0}^{l(m)} f_i(m) \mathbf{x}_i^{\top} \mathbf{T} \in \mathbb{Z}_q^{1 \times \mu}</math>  For <math>i = (m) + 1, \dots, \ell</math>:  <math>d_i = \mathbf{x}_i^{\top} \mathbf{t} \in \mathbb{Z}_q; \mathbf{D}_i = \mathbf{x}_i^{\top} \mathbf{T} \in \mathbb{Z}_q^{1 \times \mu}</math>  Return <math>([t]_2, [u]_2, [\mathbf{T}]_2, [\mathbf{u}]_2, ([d_i]_2, [\mathbf{D}_i]_2)_{(m)+1 \leq i \leq \ell})</math></p>	<p><b>CHAL(m*):</b> // one query  <math>h \leftarrow_{\\$} \mathbb{Z}_q</math>  <math>\mathbf{h}_0 = \sum f_i(m_i^*) \mathbf{x}_i \cdot h \in \mathbb{Z}_q^n; h_1 = x'_0 \cdot h \in \mathbb{Z}_q</math>  <math>(\mathbf{h}_0, h_1) \leftarrow_{\\$} \mathbb{Z}_q^n \times \mathbb{Z}_q</math>  Return <math>([h]_1, [\mathbf{h}_0]_1, [h_1]_T)</math></p> <p><b>FINALIZE(<math>\beta \in \{0, 1\}</math>):</b>  Return <math>\beta \wedge (\text{Prefix}(m^*) \cap \mathcal{Q}_{\mathcal{M}} = \emptyset)</math></p>
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**Figure 14:** Games  $\text{APR-CMA}_{\text{real}}$  and  $\text{APR-CMA}_{\text{rand}}$  for defining APR-CMA security.

an asymmetric pairing group such that  $(\mathbb{G}_2, g_2, q)$  is contained in  $\text{par}$ . Consider the games from Figure 14, where the (publicly known)  $\mu$  is defined as the rank of matrix  $\mathbf{B}$  output by  $\text{Gen}_{\text{MAC}}(\text{par})$ .

**Definition 5.8** An affine MAC over  $\mathbb{Z}_q^n$  is APR-CMA-secure if for PPT  $\mathcal{A}$ ,  $\text{Adv}_{\text{MAC}}^{\text{apr-cma}}(\mathcal{A}) := \Pr[\text{APR-CMA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] - \Pr[\text{APR-CMA}_{\text{rand}}^{\mathcal{A}} \Rightarrow 1]$  is negligible.

Unfortunately,  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$  is unlikely to be APR-CMA-secure, since in the security proof the value  $\mathbf{u}$  is not pseudorandom and, thus, we can not make  $\mathbf{h}_0$  random. The following theorem states that  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  from Section 3 with message space  $\mathcal{M} = \mathcal{B}^{\leq m} = (\mathbb{Z}_q^*)^{\leq m}$  is anonymous and delegatable.

**Theorem 5.9** Under the  $\mathcal{D}_k$ -MDDH assumption,  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  is APR-CMA-secure. In particular, for all adversaries  $\mathcal{A}$  there exists an adversary  $\mathcal{D}$  with  $\mathbf{T}(\mathcal{A}) \approx \mathbf{T}(\mathcal{D})$  and  $\text{Adv}_{\text{MAC}_{\text{HPS}}[\mathcal{D}_k]}^{\text{apr-cma}}(\mathcal{A}) \leq 2Q(\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{D}) + 1/q)$ , where  $Q$  is the maximal number of queries to  $\text{EVAL}(\cdot)$ .

The proof is postponed to Section A.3.

**THE ANONYMITY-PRESERVING TRANSFORMATION.** Let  $\mathcal{D}_k$  be a matrix distribution that outputs matrices  $\mathbf{A} \in \mathbb{Z}_q^{(k+1) \times k}$ . Let MAC be an delegatable affine MAC over  $\mathbb{Z}_q^n$  with message space  $\mathcal{M} = \mathcal{B}^{\leq m}$ . Our  $\text{AHIBKEM}[\text{MAC}, \mathcal{D}_k] = (\text{Gen}, \text{USKGen}, \text{USKDel}, \text{Enc}, \text{Dec})$  for key-space  $\mathcal{K} = \mathbb{G}_T$  and hierarchical identity space  $\mathcal{ID} = \mathcal{M} = \mathcal{B}^{\leq m}$  is defined as in Fig. 15. Compared to the HIBE construction from Section 5.4, the new construction uses a different usk rerandomization method:  $\text{USKGen}$  outputs a random basis  $\mathbf{T}$  for vector  $\mathbf{t}$  which allows rerandomization of  $\mathbf{t}$ ; similarly,  $\mathbf{u}$  and  $\mathbf{V}$  are bases for randomizing  $u$  and  $\mathbf{v}$ . Further,  $\text{Gen}$  will never return  $[\mathbf{x}_i^{\top} \mathbf{B}]_2$  and  $[\mathbf{Y}_i \mathbf{B}]_2$ , which is the key to preserve anonymity.

Correctness of  $\text{AHIBKEM}[\text{MAC}, \mathcal{D}_k]$  follows by the same argument as  $\text{HIBKEM}[\text{MAC}, \mathcal{D}_k]$ . The following theorem shows PR-HID-CPA security of our construction. Its proof is the same as the one of Theorem 5.7 except that we make the ciphertext to be random based on the APR-CMA security. Details are postponed to Appendix B.2.

**Theorem 5.10** If MAC is APR-CMA-secure and the  $\mathcal{D}_k$ -MDDH assumption holds in  $\mathbb{G}_1$  then  $\text{AHIBKEM}[\text{MAC}, \mathcal{D}_k]$  is PR-HID-CPA secure. In particular, for all adversaries  $\mathcal{A}$  there exist adversaries  $\mathcal{B}_1$  and  $\mathcal{B}_2$  with  $\mathbf{T}(\mathcal{B}_1) \approx \mathbf{T}(\mathcal{A}) \approx \mathbf{T}(\mathcal{B}_2)$  and  $\text{Adv}_{\text{AHIBKEM}[\text{MAC}, \mathcal{D}_k]}^{\text{pr-hid-cpa}}(\mathcal{A}) \leq \text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}_1) + \text{Adv}_{\text{MAC}}^{\text{apr-cma}}(\mathcal{B}_2)$ .

<p><b>Gen(par):</b>  <math>\mathbf{A} \leftarrow_{\\$} \mathcal{D}_k</math>  <math>\text{sk}_{\text{MAC}} = (\mathbf{B}, \mathbf{x}_0, \dots, \mathbf{x}_\ell, x'_0) \leftarrow_{\\$} \text{Gen}_{\text{MAC}}(\text{par})</math>  For <math>i = 0, \dots, \ell</math>:  <math>\mathbf{Y}_i \leftarrow_{\\$} \mathbb{Z}_q^{k \times n}; \mathbf{Z}_i = (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \cdot \mathbf{A} \in \mathbb{Z}_q^{n \times k}</math>  <math>\mathbf{y}'_0 \leftarrow_{\\$} \mathbb{Z}_q^k; \mathbf{z}'_0 = (\mathbf{y}'_0^\top \mid x'_0) \cdot \mathbf{A} \in \mathbb{Z}_q^{1 \times k}</math>  <math>\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, [\mathbf{z}'_0]_1)</math>  <math>\text{sk} := (\text{sk}_{\text{MAC}}, (\mathbf{Y}_i)_{0 \leq i \leq \ell}, \mathbf{y}'_0)</math>  Return <math>(\text{pk}, \text{sk})</math></p> <p><b>USKGen(sk, id <math>\in \mathcal{ID}</math>):</b>  <math>([\mathbf{t}]_2, [\mathbf{u}]_2) \leftarrow_{\\$} \text{Tag}(\text{sk}_{\text{MAC}}, \text{id})</math>  // <math>\mathbf{t} \in \mathbb{Z}_q^n; \mathbf{u} = \sum f_i(\text{id}) \mathbf{x}_i^\top \mathbf{t} + x'_0 \in \mathbb{Z}_q</math>  <math>\mathbf{v} = \sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Y}_i \mathbf{t} + \mathbf{y}'_0 \in \mathbb{Z}_q^k</math>  <math>\mathbf{S} \leftarrow_{\\$} \mathbb{Z}_q^{n \times \mu}; \mathbf{T} = \mathbf{B} \cdot \mathbf{S} \in \mathbb{Z}_q^{n \times \mu}</math>  <math>\mathbf{u} = \sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{x}_i^\top \mathbf{T} \in \mathbb{Z}_q^{1 \times \mu}</math>  <math>\mathbf{V} = \sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Y}_i \mathbf{T} \in \mathbb{Z}_q^{k \times \mu}</math>  For <math>i = l(\text{id}) + 1, \dots, \ell</math>:  <math>d_i = \mathbf{x}_i^\top \mathbf{t} \in \mathbb{Z}_q; \mathbf{D}_i = \mathbf{x}_i^\top \mathbf{T} \in \mathbb{Z}_q^{1 \times \mu}</math>  <math>\mathbf{e}_i = \mathbf{Y}_i \mathbf{t} \in \mathbb{Z}_q^k; \mathbf{E}_i = \mathbf{Y}_i \mathbf{T} \in \mathbb{Z}_q^{k \times \mu}</math>  <math>\text{usk}[\text{id}] := ([\mathbf{t}]_2, [\mathbf{u}]_2, [\mathbf{v}]_2) \in \mathbb{G}_2^n \times \mathbb{G}_2^1 \times \mathbb{G}_2^k</math>  <math>\text{udk}[\text{id}] := ([\mathbf{T}]_2, [\mathbf{u}]_2, [\mathbf{V}]_2, ([d_i]_2, [\mathbf{D}_i]_2, [\mathbf{e}_i]_2, [\mathbf{E}_i]_2)_{l(\text{id}) &lt; i \leq \ell})</math>  <math>\in \mathbb{G}_2^{n \times \mu} \times \mathbb{G}_2^{1 \times \mu} \times \mathbb{G}_2^{k \times \mu} \times (\mathbb{G}_2 \times \mathbb{G}_2^{1 \times \mu} \times \mathbb{G}_2^k \times \mathbb{G}_2^{k \times \mu})^{\ell - l(\text{id})}</math>  Return <math>(\text{usk}[\text{id}], \text{udk}[\text{id}])</math></p> <p><b>Enc(pk, id):</b>  <math>\mathbf{r} \leftarrow_{\\$} \mathbb{Z}_q^k</math>  <math>\mathbf{c}_0 = \mathbf{A} \mathbf{r} \in \mathbb{Z}_q^{k+1}</math>  <math>\mathbf{c}_1 = (\sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Z}_i) \cdot \mathbf{r} \in \mathbb{Z}_q^n</math>  <math>K = \mathbf{z}'_0 \cdot \mathbf{r} \in \mathbb{Z}_q</math>  Return <math>\mathbf{K} = [K]_T</math> and <math>\mathbf{C} = ([\mathbf{c}_0]_1, [\mathbf{c}_1]_1)</math></p>	<p><b>USKDel(usk[id], udk[id], id <math>\in \mathcal{B}^p</math>, id<sub>p+1</sub> <math>\in \mathcal{B}</math>):</b>  If <math>p \geq m</math>, then return <math>\perp</math>  <math>\text{id}' := (\text{id}_1, \dots, \text{id}_p, \text{id}_{p+1}) \in \mathcal{B}^{p+1}</math>  // Delegation of <math>(u, \mathbf{v})</math> and <math>(\mathbf{u}, \mathbf{V})</math>:  <math>\hat{u} = u + \sum_{i=l(\text{id})+1}^{l(\text{id}')} f_i(\text{id}') d_i \in \mathbb{Z}_q</math>  <math>\hat{\mathbf{v}} = \mathbf{v} + \sum_{i=l(\text{id})+1}^{l(\text{id}')} f_i(\text{id}') \mathbf{e}_i \in \mathbb{Z}_q^k</math>  <math>\hat{\mathbf{u}} = \mathbf{u} + \sum_{i=l(\text{id})+1}^{l(\text{id}')} f_i(\text{id}') \mathbf{D}_i \in \mathbb{Z}_q^{1 \times \mu}</math>  <math>\hat{\mathbf{V}} = \mathbf{V} + \sum_{i=l(\text{id})+1}^{l(\text{id}')} f_i(\text{id}') \mathbf{E}_i \in \mathbb{Z}_q^{k \times \mu}</math>  // Rerandomization of <math>(\hat{u}, \hat{\mathbf{v}})</math> and <math>(\hat{\mathbf{u}}, \hat{\mathbf{V}})</math>:  <math>\mathbf{s}' \leftarrow_{\\$} \mathbb{Z}_q^\mu; \mathbf{S} \leftarrow_{\\$} \mathbb{Z}_q^{n \times \mu}</math>  <math>\mathbf{t}' = \mathbf{t} + \mathbf{T} \mathbf{s}' \in \mathbb{Z}_q^n; \mathbf{T}' = \hat{\mathbf{T}} \cdot \mathbf{S}' \in \mathbb{Z}_q^{n \times \mu}</math>  <math>\mathbf{u}' = \hat{u} + \hat{\mathbf{u}} \cdot \mathbf{s}' \in \mathbb{Z}_q; \mathbf{u}' = \hat{\mathbf{u}} \cdot \mathbf{S}' \in \mathbb{Z}_q^{1 \times \mu}</math>  <math>\mathbf{v}' = \hat{\mathbf{v}} + \hat{\mathbf{V}} \cdot \mathbf{s}' \in \mathbb{Z}_q^k; \mathbf{V}' = \hat{\mathbf{V}} \cdot \mathbf{S}' \in \mathbb{Z}_q^{k \times \mu}</math>  // Rerandomization of <math>d'_i</math> and <math>\mathbf{e}'_i</math>:  For <math>i = l(\text{id}') + 1, \dots, \ell</math>:  <math>d'_i = d_i + \mathbf{D}_i \mathbf{s}' \in \mathbb{Z}_q; \mathbf{D}'_i = \mathbf{D}_i \cdot \mathbf{S}' \in \mathbb{Z}_q^{1 \times \mu}</math>  <math>\mathbf{e}'_i = \mathbf{e}_i + \mathbf{E}_i \mathbf{s}' \in \mathbb{Z}_q^k; \mathbf{E}'_i = \mathbf{E}_i \cdot \mathbf{S}' \in \mathbb{Z}_q^{k \times \mu}</math>  <math>\text{usk}[\text{id}'] := ([\mathbf{t}']_2, [\mathbf{u}']_2, [\mathbf{v}']_2)</math>  <math>\text{udk}[\text{id}'] := ([\mathbf{T}']_2, [\mathbf{u}']_2, [\mathbf{V}']_2, ([d'_i]_2, [\mathbf{D}'_i]_2, [\mathbf{e}'_i]_2, [\mathbf{E}'_i]_2)_{l(\text{id}') &lt; i \leq \ell})</math>  Return <math>(\text{usk}[\text{id}'], \text{udk}[\text{id}'])</math></p> <p><b>Dec(usk[id], id, C):</b>  Parse <math>\text{usk}[\text{id}] = ([\mathbf{t}]_2, [\mathbf{u}]_2, [\mathbf{v}]_2)</math>  Parse <math>\mathbf{C} = ([\mathbf{c}_0]_1, [\mathbf{c}_1]_1)</math>  <math>\mathbf{K} = e([\mathbf{c}_0]_1, \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix}_2) \cdot e([\mathbf{c}_1]_1, [\mathbf{t}]_2)^{-1}</math>  Return <math>\mathbf{K} \in \mathbb{G}_T</math></p>
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**Figure 15:** Definition of the transformation AHIBKEM[MAC,  $\mathcal{D}_k$ ].  $\mu$  denotes the rank of  $\mathbf{B}$ .

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<b>INITIALIZE:</b> // Games $G_0, \overline{G_{1,i}}, G_2$ $\mathbf{A} \leftarrow \mathcal{D}_k; \mathbf{B} := \overline{\mathbf{A}}$ For $1 \leq j \leq m$ and $j' = 0, 1$ : $\mathbf{x}_{j,j'} \leftarrow_{\mathcal{S}} \mathbb{Z}_q^k$ $x'_0 \leftarrow_{\mathcal{S}} \mathbb{Z}_q$ ; $[x'_0]$ is undefined Return $([\mathbf{B}]_2, ([\mathbf{x}_{j,j'}^\top \mathbf{B}]_2)_{1 \leq j \leq m, j'=0,1})$  <b>CHAL(<math>\mathbf{m}^*</math>):</b> // Games $G_0, \overline{G_{1,i}}, \overline{G_{2,i}}$ one query $[x'_0] = \text{RF}_i(\mathbf{m}_{ i}^*)$ $h \leftarrow_{\mathcal{S}} \mathbb{Z}_q$ ; $\mathbf{h}_0 = \sum \mathbf{x}_{j,m_j} h \in \mathbb{Z}_q^k$ ; $h_1 = x'_0 h \in \mathbb{Z}_q$ ; $[h_1] \leftarrow_{\mathcal{S}} \overline{\mathbb{Z}_q}$ Return $([h]_1, [\mathbf{h}_0]_1, [h_1]_T)$	<b>EVAL(<math>\mathbf{m}</math>):</b> // Games $G_0, \overline{G_{1,i}}, \overline{G_{2,i}}$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ $\mathbf{s} \leftarrow_{\mathcal{S}} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s} \in \mathbb{Z}_q^k$ $u = \sum_{j=1}^{ \mathbf{m} } \mathbf{x}_{j,m_j}^\top \mathbf{t} + x'_0$ $[u] = \sum_{j=1}^{ \mathbf{m} } \mathbf{x}_{j,m_j}^\top \mathbf{t} + \text{RF}_i(\mathbf{m}_{ i})$ $[u] \leftarrow_{\mathcal{S}} \overline{\mathbb{Z}_q}$ For $ \mathbf{m}  < j \leq m$ and $j' = 0, 1$ : $d_{j,j'} = \mathbf{x}_{j,j'}^\top \mathbf{t} \in \mathbb{Z}_q$ Return $([\mathbf{t}]_2, [u]_2, ([d_{j,j'}]_2)_{ \mathbf{m}  < j \leq m, j'=0,1})$  <b>FINALIZE(<math>d \in \{0, 1\}</math>):</b> // Games $G_0$ - $G_2$ Return $(\text{Prefix}(\mathbf{m}^*) \cap \mathcal{Q}_{\mathcal{M}} = \emptyset) \wedge d$
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**Figure 16:** Games  $G_0, G_{1,i}$  ( $0 \leq i \leq m$ ) and  $G_2$  for the proof of Theorem 5.3.

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## A Security Proofs for the MACs

### A.1 Security of Delegatable $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$ (Theorem 5.3)

We prove Theorem 5.3 by defining a sequence of intermediate games  $G_0$ - $G_2$  as in Figure 16.

Similar to Lemma 3.4 and 3.4, we obtain the following lemma.



<p><b>INITIALIZE:</b>  <math>b \leftarrow_{\mathcal{S}} \{0, 1, \perp\}</math>  If <math>b = 0</math> or <math>b = 1</math>:  <math>\mathbf{r}_{1-b} \leftarrow_{\mathcal{S}} \mathbb{Z}_q^{k+1}; \mathbf{x}_{i,1-b}^{\top} \bar{\mathbf{A}} := \mathbf{r}_{1-b}^{\top} \mathbf{A} \in \mathbb{Z}_q^{1 \times k}</math>  Else for <math>j = 0, 1</math>:  <math>\mathbf{r}_j \leftarrow_{\mathcal{S}} \mathbb{Z}_q^{k+1}; \mathbf{x}_{i,j}^{\top} \bar{\mathbf{A}} := \mathbf{r}_j^{\top} \mathbf{A} \in \mathbb{Z}_q^{1 \times k}</math>    For <math>1 \leq j \leq m</math> and <math>j' = 0, 1</math>:  if <math>j \neq i</math> or <math>j' = b</math> then <math>\mathbf{x}_{j,j'} \leftarrow_{\mathcal{S}} \mathbb{Z}_q^k</math>  Return <math>([\bar{\mathbf{A}}]_2, ([\mathbf{x}_{j,j'}^{\top} \bar{\mathbf{A}}]_2)_{1 \leq j \leq m, j'=0,1})</math>    <b>CHAL</b>(<math>\mathbf{m}^*</math>):  Abort if <math>\mathbf{m}_i^* \neq b</math>  <math>h \leftarrow_{\mathcal{S}} \mathbb{Z}_q; x'_0 = \text{RF}_i(\mathbf{m}_i^*)</math>  <math>\mathbf{h}_0 = (\sum_{j=1}^{ \mathbf{m}^* } \mathbf{x}_{j,m_j^*})h \in \mathbb{Z}_q^k; h_1 = x'_0 h \in \mathbb{Z}_q</math>  Return <math>([h]_1, [\mathbf{h}_0]_1, [h_1]_T)</math></p>	<p><b>EVAL</b>(<math>\mathbf{m}</math>):  <math>\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}; c = \alpha_i(\mathbf{m}_i)</math>  <math>\mathbf{s}' \leftarrow_{\mathcal{S}} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{A}\mathbf{s}' + \bar{\mathbf{H}}_c</math>  <math>u = \begin{cases} (\sum_{j=1}^{ \mathbf{m} } \mathbf{x}_{j,m_j}^{\top} \mathbf{t} + \text{RF}_{i-1}(\mathbf{m}_{ i-1})) &amp; \mathbf{m}_i \in \{b, \perp\} \\ (\sum_{j \neq i} \mathbf{x}_{j,m_j}^{\top} \mathbf{t} + \mathbf{r}^{\top}(\mathbf{A}\mathbf{s}' + \bar{\mathbf{H}}_c) + \text{RF}_{i-1}(\mathbf{m}_{ i-1})) &amp; \mathbf{m}_i = 1 - b \end{cases}</math>  <math>d_{j,j'} = \mathbf{x}_{j,j'}^{\top} \mathbf{t} \text{ } ( \mathbf{m}  &lt; j \leq m \text{ and } j' = 0, 1)</math>  Return <math>([\mathbf{t}]_2, [u]_2, ([d_{j,j'}]_2)_{ \mathbf{m}  &lt; j \leq m, j'=0,1})</math>    <b>FINALIZE</b>(<math>d \in \{0, 1\}</math>):  Return <math>(\text{Prefix}(\mathbf{m}^*) \cap \mathcal{Q}_{\mathcal{M}} = \emptyset) \wedge d</math></p>
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**Figure 17:** Description of  $\mathcal{B}'(\mathcal{G}, [\mathbf{A}]_2, [\mathbf{H}]_2)$  interpolating between the Games  $\mathbf{G}_{1,i}$  and  $\mathbf{G}_{1,i-1}$ , where  $\bar{\mathbf{H}}_c$  denotes the  $c$ -th column of  $\bar{\mathbf{H}}$  and  $\alpha_i : \{0, 1\}^i \rightarrow \{1, \dots, Q\}$  is an injective function.

**Lemma A.1**  $\Pr[\text{HPR-CMA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_0^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_{1,0}^{\mathcal{A}} \Rightarrow 1]$ .

**Lemma A.2** *There exists an adversary  $\mathcal{B}$  with  $\mathbf{T}(\mathcal{B}) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}) - \frac{1}{q-1} \geq \frac{1}{3} |\Pr[\mathbf{G}_{1,i}^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathbf{G}_{1,i-1}^{\mathcal{A}} \Rightarrow 1]|$ .*

**Proof:** We build an adversary  $\mathcal{B}'$  against the  $Q$ -fold  $\mathcal{D}_k$ -MDDH Assumption such that

$$\text{Adv}_{\mathcal{D}_k, \text{GGen}}^Q(\mathcal{B}') \geq \frac{1}{3} |\Pr[\mathbf{G}_{1,i}^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathbf{G}_{1,i-1}^{\mathcal{A}} \Rightarrow 1]|, \quad (5)$$

which implies the lemma by the random self reducibility of the MDDH assumption (Lemma 2.3).

For any message  $\mathbf{m} \in \mathcal{M} := \{0, 1\}^{\leq m}$ , define  $\mathbf{m}_i := \perp$  if  $|\mathbf{m}| < i$ . On input a  $\mathcal{D}_k$ -MDDH challenge  $([\mathbf{A}]_2, [\mathbf{H}]_2) \in \mathbb{G}_2^{(k+1) \times k} \times \mathbb{G}_2^{(k+1) \times Q}$ ,  $\mathcal{B}'$  first picks a random value  $b \in \{0, 1, \perp\}$  which is a guess for  $\mathbf{m}_i^*$  and defines  $\text{RF}_i : \{0, 1\}^i \rightarrow \mathbb{Z}_q$  as

$$\text{RF}_i(\mathbf{m}_{|i}) = \begin{cases} \text{RF}_{i-1}(\mathbf{m}_{|i-1}) & \mathbf{m}_i = b \text{ or } \mathbf{m}_i = \perp \\ \text{RF}_{i-1}(\mathbf{m}_{|i-1}) + \text{RF}'_{i-1}(\mathbf{m}_{|i-1}) & \text{otherwise} \end{cases}, \quad (6)$$

The definition of  $\mathcal{B}'$  is given in Figure 17. Note that if  $\text{RF}_{i-1}$  and  $\text{RF}'_{i-1}$  are random functions, then  $\text{RF}_i(\mathbf{m}_{|i-1}||b)$  and  $\text{RF}_i(\mathbf{m}_{|i-1}||1-b)$  are independent and both uniformly random.

Assume  $\mathcal{B}'$  correctly guesses  $b = \mathbf{m}_i^*$  (which happens with probability  $1/3$ ). By the definition of  $\text{RF}_i$  and by  $\mathbf{m}_i^* = b$  we have  $\text{RF}_i(\mathbf{m}_{|i}^*) = \text{RF}_{i-1}(\mathbf{m}_{|i-1}^*)$ , which implies  $\text{CHAL}(\mathbf{m}^*)$  is identically distributed in  $\mathbf{G}_{1,i}$  and  $\mathbf{G}_{1,i-1}$ .

We now analyze the output distribution of the EVAL queries. First note that  $\mathbf{t}$  is uniformly random over  $\mathbb{Z}_q^k$  in both games  $\mathbf{G}_{1,i}$  and  $\mathbf{G}_{1,i-1}$ . As for the distribution of  $u$ , we only need to consider the case  $\mathbf{m}_i = 1 - b$ , since  $u$  for  $\mathbf{m}_i \in \{b, \perp\}$  is identically distributed in games  $\mathbf{G}_{1,i}$  and  $\mathbf{G}_{1,i-1}$ . Assume  $\mathbf{m}_i = 1 - b$ . Write  $\bar{\mathbf{H}}_c = \mathbf{A}\mathbf{W}_c + \mathbf{R}_c$  for some  $\mathbf{W}_c \in \mathbb{Z}_q^k$ , where  $\mathbf{R}_c = 0$  (i.e.,  $\bar{\mathbf{H}}$  is from the  $\mathcal{D}_k$ -MDDH distribution) or  $\mathbf{R}_c$  is uniform.

$$\begin{aligned}
u &= \sum_{j \neq i} \mathbf{x}_{j,m_j}^\top \mathbf{t} + \mathbf{r}^\top \mathbf{A}(\mathbf{s}' + \mathbf{W}_c) + \mathbf{r}^\top \mathbf{R}_c + \text{RF}_{i-1}(\mathbf{m}_{|i-1}) \\
&= \sum_{j \neq i} \mathbf{x}_{j,m_j}^\top \mathbf{t} + \mathbf{x}_{i,1-b}^\top \underbrace{\bar{\mathbf{A}}(\mathbf{s}' + \mathbf{W}_c)}_{\mathbf{t}} + \mathbf{r}^\top \mathbf{R}_c + \text{RF}_{i-1}(\mathbf{m}_{|i-1}) \\
&= \sum_{j=0}^{|\mathbf{m}|} \mathbf{x}_{j,m_j}^\top \mathbf{t} + \mathbf{r}^\top \mathbf{R}_c + \text{RF}_{i-1}(\mathbf{m}_{|i-1}).
\end{aligned}$$

If  $\mathbf{R}_c = 0$ , then  $u$  is distributed as in game  $\mathbf{G}_{1,i-1}$ . If  $\mathbf{R}_c$  is uniform, then define  $\text{RF}'(\mathbf{m}_{|i-1}) := \mathbf{r}^\top \mathbf{R}_c$  and  $u$  is distributed as in  $\mathbf{G}_{1,i}$ .  $\blacksquare$

**Lemma A.3**  $\Pr[\mathbf{G}_{1,m}^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1]$ .

**Proof:** In  $\mathbf{G}_{1,m}$ ,  $u$  returned by the  $\text{EVAL}(\mathbf{m})$  oracle is masked by  $\text{RF}_m(\mathbf{m})$ , which is uniformly random and independent of  $\mathbf{m}$  and the secrets  $\mathbf{x}_{j,j'}$  and  $x'_0$ . Thus,  $u$  is uniformly random in game  $\mathbf{G}_{1,m}$ . Since nothing about  $x'_0$  is leaked from  $\text{EVAL}$  and  $x'_0 = \text{RF}_m(\mathbf{m}^*)$ ,  $h_1$  is distributed uniformly at random over  $\mathbb{Z}_q$ .  $\blacksquare$

In contrast to Lemma 3.6,  $\mathbf{h}_0$  is not pseudorandom here, since  $\mathcal{A}$  learns  $([\mathbf{x}_{j,j'}^\top \mathbf{B}]_2)_{1 \leq j \leq m, j'=0,1}$  from its call to  $\text{INITIALIZE}$ . By checking the pairing equation  $e([h]_1, [\sum \mathbf{x}_{j,m_j}^\top \mathbf{B}]_2) = e([\mathbf{h}_0]_1, [\mathbf{B}]_2)$ ,  $\mathcal{A}$  verifies if  $\mathbf{h}_0$  is properly computed.

Finally, we do all the previous steps in reverse order similar to Lemma 3.7, and then we end up with the following lemma.

**Lemma A.4** *There exists an adversary  $\mathcal{B}$  with  $\mathbf{T}(\mathcal{B}) \approx \mathbf{T}(\mathcal{A})$  and  $3m \text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}) - \frac{3m}{q-1} \geq |\Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1] - \Pr[\text{HPR}_0\text{-CMA}_{\text{rand}}^{\mathcal{A}} \Rightarrow 1]|$ .*

Theorem 5.3 follows by combining Lemmas A.1-A.4

## A.2 Security of Delegatable $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$ (Theorem 5.4)

We prove Theorem 5.4 by defining a sequence of intermediate games  $\mathbf{G}_0\text{-}\mathbf{G}_2$  as in Figure 18.

**Lemma A.5**  $\Pr[\text{HPR-CMA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_0^{\mathcal{A}} \Rightarrow 1]$

Similar to the proof of Theorem 3.8, we now define the games  $\mathbf{G}_{1,i}$  and  $\mathbf{G}'_{1,i}$ . By the same arguments as in Section 3.3, we have the following two lemmas:

**Lemma A.6**  $\Pr[\mathbf{G}_0^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_{1,1}^{\mathcal{A}} \Rightarrow 1]$

**Lemma A.7** *There exists an adversary  $\mathcal{B}$  with  $\mathbf{T}(\mathcal{B}) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}) \geq |\Pr[\mathbf{G}'_{1,i}^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathbf{G}_{1,i}^{\mathcal{A}} \Rightarrow 1]|$*

**Lemma A.8**  $|\Pr[\mathbf{G}_{1,i+1}^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathbf{G}'_{1,i}^{\mathcal{A}} \Rightarrow 1]| \leq 1/q$ .

**Proof:** At a high level, those two games are only separated by the 2-universality of the underlying hash proof system. The following arguments are similar to Lemma 3.11. Let  $\mathbf{m}$  the  $i$ -th queried message. We have  $\mathbf{m} \neq \mathbf{m}^*$ , so there exists an index  $i'$  such that  $\mathbf{m}_{i'} \neq \mathbf{m}_{i'}^*$ , where  $\mathbf{m}_{i'}$  (resp.  $\mathbf{m}_{i'}^*$ ) denotes the  $i'$ -th entry of  $\mathbf{m}$  (resp.  $\mathbf{m}^*$ ). By the definition of  $\text{HPR}_0\text{-CMA}$  security,  $\mathbf{m}$  can not be a prefix of  $\mathbf{m}^*$ . Thus, either  $i' \leq |\mathbf{m}^*|$ , which leads to the same proof as in Lemma 3.11, or  $i' \geq |\mathbf{m}^*| + 1$  and  $\mathbf{m}_j^* = \mathbf{m}_j$  for all  $j \leq |\mathbf{m}^*|$ . In the latter case,  $\mathcal{A}$  obtains the following equations in the unknown variables  $\mathbf{x}_0, \mathbf{x}_{i'}$  in an information-theoretical way:

<b>INITIALIZE:</b> // Games $G_0$ - $G_2$ $\mathbf{B} \leftarrow_{\$} \mathcal{D}_k; x'_0 \leftarrow_{\$} \mathbb{Z}_q$ For $j = 0, \dots, m$ : $\mathbf{x}_j \leftarrow_{\$} \mathbb{Z}_q^{k+1}$ Return $([\mathbf{B}]_2, ([\mathbf{x}_j^\top \mathbf{B}]_2)_{0 \leq j \leq m})$  <b>CHAL(<math>\mathbf{m}^*</math>):</b> // Games $G_0$ - $G_{1,Q+1}, \boxed{G_2}$ $h \leftarrow_{\$} \mathbb{Z}_q$ $\mathbf{h}_0 = (\mathbf{x}_0^\top + \sum \mathbf{m}_i^* \cdot \mathbf{x}_i^\top)h \in \mathbb{Z}_q^{k+1}$ $h_1 = x'_0 h \in \mathbb{Z}_q, [h_1 \leftarrow_{\$} \mathbb{Z}_q]$ Return $([h]_1, [\mathbf{h}_0]_1, [h_1]_T)$  <b>FINALIZE(<math>d \in \{0, 1\}</math>):</b> // Games $G_0$ - $G_2$ Return $(\text{Prefix}(\mathbf{m}^*) \cap \mathcal{Q}_{\mathcal{M}} = \emptyset) \wedge d$ .  <b>EVAL(<math>\mathbf{m}</math>):</b> // Games $G_2$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ $(\mathbf{t}, u) \leftarrow_{\$} \mathbb{Z}_q^{k+1} \times \mathbb{Z}_q$ For $j =  \mathbf{m}  + 1, \dots, m$ : $d_j = \mathbf{x}_j^\top \mathbf{t}$ Return $([\mathbf{t}]_2, [u]_2, ([d_j]_2)_{ \mathbf{m}  < j \leq m})$	<b>EVAL(<math>\mathbf{m}</math>):</b> // Games $G_0$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s}$ $u = (\mathbf{x}_0^\top + \sum_{j=1}^{ \mathbf{m} } \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0$ For $ \mathbf{m}  < j \leq m$ : $d_j = \mathbf{x}_j^\top \mathbf{t}$ Return $([\mathbf{t}]_2, [u]_2, ([d_j]_2)_{ \mathbf{m}  < j \leq m})$  <b>EVAL(<math>\mathbf{m}</math>):</b> // Games $G_{1,i}, \boxed{G'_{1,i}}$ $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}$ // Let $\mathbf{m}$ be the $c$ -th query ( $1 \leq c \leq Q$ ) If $c < i$ then $(\mathbf{t}, u) \leftarrow_{\$} \mathbb{Z}_q^{k+1} \times \mathbb{Z}_q$ If $c > i$ then $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s}$ $u = (\mathbf{x}_0^\top + \sum_{j=1}^{ \mathbf{m} } \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0$ If $c = i$ then $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s}; [\mathbf{t} \leftarrow_{\$} \mathbb{Z}_q^{k+1}]$ $u = (\mathbf{x}_0^\top + \sum_{j=1}^{ \mathbf{m} } \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0$ For $j =  \mathbf{m}  + 1, \dots, m$ : $d_j = \mathbf{x}_j^\top \mathbf{t}$ Return $([\mathbf{t}]_2, [u]_2, ([d_j]_2)_{ \mathbf{m}  < j \leq m})$
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**Figure 18:** Games  $G_0$  to  $G_2$  for the proof of Theorem 5.4.

$$\begin{pmatrix} \mathbf{B}^\top \mathbf{x}_0 \\ \mathbf{B}^\top \mathbf{x}_{i'} \\ \mathbf{h}_0 \\ u - x'_0 \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{B}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^\top \\ h \cdot \mathbf{I}_{k+1} & \mathbf{0} \\ \mathbf{t}^\top & \mathbf{m}_{i'} \mathbf{t}^\top \end{pmatrix}}_{=: \mathbf{M}} \cdot \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_{i'} \end{pmatrix},$$

where  $\mathbf{B}^\top \mathbf{x}_0$  and  $\mathbf{B}^\top \mathbf{x}_{i'}$  are from INITIALIZE,  $\mathbf{h}_0$  is from CHAL( $\mathbf{m}^*$ ), and  $u - x'_0$  is from EVAL( $\mathbf{m}$ ).  $\mathbf{I}_{k+1}$  denotes the  $(k+1) \times (k+1)$  identity matrix.

As shown in Lemma 3.11, the last row of  $\mathbf{M} \in \mathbb{Z}_q^{(3k+2) \times (2k+2)}$  is linearly independent of the first  $2k$  rows (except with probability  $1/q$ ). Also, the last row is linearly independent of rows  $2k+1$  to  $3k+1$ . Thus,  $u - x'_0$  is linearly independent of  $\mathbf{B}^\top \mathbf{x}_0$ ,  $\mathbf{B}^\top \mathbf{x}_{i'}$  and  $\mathbf{h}_0$  and therefore  $u$  is distributed uniformly at random in  $G'_{1,i}$ . ■

**Lemma A.9**  $\Pr[G_2^A \Rightarrow 1] = \Pr[G_{1,Q+1}^A \Rightarrow 1]$ .

**Proof:** In  $G_2$ , we replace  $h_1$  output by CHAL( $\mathbf{m}^*, c$ ) with a random value. Similar to Lemma 3.12, since all the answers of EVAL are random and independent of  $x'_0$ ,  $h_1$  is uniformly random in the view of  $\mathcal{A}$ . ■

We apply all the arguments before in a reverse order and then we easily get the following:

**Lemma A.10** *There exists an adversary  $\mathcal{B}$  with  $\mathbf{T}(\mathcal{B}) \approx \mathbf{T}(\mathcal{A})$  and  $|\Pr[\text{HPR}_0\text{-CMA}_{\text{rand}}^A \Rightarrow 1] - \Pr[G_2^A \Rightarrow 1]| \leq Q(\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}) + 1/q)$ .*

The proof of the theorem follows by combining Lemmas A.5-A.10.

### A.3 Anonymity of delegatable $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$ (Theorem 5.9)

We prove Theorem 5.9 by defining a sequence of intermediate games as in Figure 19. Let  $\mathcal{A}$  be an adversary against the APR-CMA-security of  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$ .

$G_0$  is the real attack game and we have:

<p><b>INITIALIZE:</b> // Games <math>G_0</math>-<math>G_2</math></p> <p><math>\mathbf{B} \leftarrow_{\\$} \mathcal{D}_k; x'_0 \leftarrow_{\\$} \mathbb{Z}_q</math></p> <p>For <math>j = 0, \dots, \ell : \mathbf{x}_j \leftarrow_{\\$} \mathbb{Z}_q^{k+1}</math></p> <p>Return <math>\varepsilon</math></p> <p><b>CHAL(<math>\mathbf{m}^*</math>):</b> // Games <math>G_0</math>-<math>G_{1,Q+1}, \boxed{G_2}</math></p> <p><math>h \leftarrow_{\\$} \mathbb{Z}_q</math></p> <p><math>\mathbf{h}_0 = (\mathbf{x}_0 + \sum \mathbf{m}_j^* \cdot \mathbf{x}_j)h \in \mathbb{Z}_q^{k+1}; \boxed{\mathbf{h}_0 \leftarrow_{\\$} \mathbb{Z}_q^{k+1}}</math></p> <p><math>h_1 = x'_0 h \in \mathbb{Z}_q; \boxed{h_1 \leftarrow_{\\$} \mathbb{Z}_q}</math></p> <p>Return <math>([h]_1, [\mathbf{h}_0]_1, [h_1]_T)</math></p> <p><b>FINALIZE(<math>d \in \{0, 1\}</math>):</b> // Games <math>G_0</math>-<math>G_2</math></p> <p>Return <math>d \wedge (\text{Prefix}(\mathbf{m}^*) \cap \mathcal{Q}_{\mathcal{M}} = \emptyset)</math></p> <p><b>EVAL(<math>\mathbf{m}</math>):</b> // Game <math>G_2</math></p> <p><math>\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}</math></p> <p><math>(\mathbf{t}, u) \leftarrow_{\\$} \mathbb{Z}_q^{k+1} \times \mathbb{Z}_q</math></p> <p><math>(\mathbf{T}, \mathbf{u}) \leftarrow_{\\$} \mathbb{Z}_q^{(k+1) \times k} \times \mathbb{Z}_q^{1 \times k}</math></p> <p>For <math>j =  \mathbf{m}  + 1, \dots, m : d_j = \mathbf{x}_j^\top \mathbf{t}; \mathbf{D}_j = \mathbf{x}_j^\top \mathbf{T}</math></p> <p>Return <math>([\mathbf{t}]_2, [u]_2, [\mathbf{T}]_2, [\mathbf{u}]_2, ([d_j]_2, [\mathbf{D}_j]_2)_{ \mathbf{m}  &lt; j \leq m})</math></p>	<p><b>EVAL(<math>\mathbf{m}</math>):</b> // Games <math>G_0</math></p> <p><math>\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}</math></p> <p><math>\mathbf{s} \leftarrow_{\\$} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s} \in \mathbb{Z}_q^{k+1}</math></p> <p><math>u = (\mathbf{x}_0^\top + \sum_{j=1}^{ \mathbf{m} } \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0 \in \mathbb{Z}_q</math></p> <p><math>\mathbf{S} \leftarrow_{\\$} \mathbb{Z}_q^{k \times k}; \mathbf{T} = \mathbf{B} \cdot \mathbf{S} \in \mathbb{Z}_q^{(k+1) \times k}</math></p> <p><math>\mathbf{u} = (\mathbf{x}_0^\top + \sum \mathbf{m}_j \mathbf{x}_j^\top) \mathbf{T} \in \mathbb{Z}_q^{1 \times k}</math></p> <p>For <math> \mathbf{m}  &lt; j \leq m : d_j = \mathbf{x}_j^\top \mathbf{t} \in \mathbb{Z}_q; \mathbf{D}_j = \mathbf{x}_j^\top \mathbf{T} \in \mathbb{Z}_q^{1 \times k}</math></p> <p>Return <math>([\mathbf{t}]_2, [u]_2, [\mathbf{T}]_2, [\mathbf{u}]_2, ([d_j]_2, [\mathbf{D}_j]_2)_{ \mathbf{m}  &lt; j \leq m})</math></p> <p><b>EVAL(<math>\mathbf{m}</math>):</b> // Games <math>G_{1,i}, \boxed{G'_{1,i}}</math></p> <p><math>\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}} \cup \{\mathbf{m}\}</math> // Let <math>\mathbf{m}</math> be the <math>c</math>-th query (<math>1 \leq c \leq Q</math>)</p> <p>If <math>c &lt; i</math> then</p> <p><math>(\mathbf{t}, u) \leftarrow_{\\$} \mathbb{Z}_q^{k+1} \times \mathbb{Z}_q; (\mathbf{T}, \mathbf{u}) \leftarrow_{\\$} \mathbb{Z}_q^{(k+1) \times k} \times \mathbb{Z}_q^{1 \times k}</math></p> <p>If <math>c &gt; i</math> then</p> <p><math>\mathbf{s} \leftarrow_{\\$} \mathbb{Z}_q^k, \mathbf{t} = \mathbf{B}\mathbf{s}</math></p> <p><math>u = (\mathbf{x}_0^\top + \sum_{j=1}^{ \mathbf{m} } \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0</math></p> <p><math>\mathbf{S} \leftarrow_{\\$} \mathbb{Z}_q^{k \times k}; \mathbf{T} = \mathbf{B} \cdot \mathbf{S}</math></p> <p><math>\mathbf{u} = (\mathbf{x}_0^\top + \sum \mathbf{m}_j \mathbf{x}_j^\top) \mathbf{T}</math></p> <p>If <math>c = i</math> then</p> <p><math>\mathbf{s} \leftarrow_{\\$} \mathbb{Z}_q^k; \mathbf{t} = \mathbf{B}\mathbf{s}; \boxed{\mathbf{t} \leftarrow_{\\$} \mathbb{Z}_q^{k+1}}</math></p> <p><math>u = (\mathbf{x}_0^\top + \sum_{j=1}^{ \mathbf{m} } \mathbf{m}_j \cdot \mathbf{x}_j^\top) \mathbf{t} + x'_0</math></p> <p><math>\mathbf{S} \leftarrow_{\\$} \mathbb{Z}_q^{k \times k}; \mathbf{T} = \mathbf{B} \cdot \mathbf{S}; \boxed{\mathbf{T} \leftarrow_{\\$} \mathbb{Z}_q^{(k+1) \times k}}</math></p> <p><math>\mathbf{u} = (\mathbf{x}_0^\top + \sum \mathbf{m}_j \mathbf{x}_j^\top) \mathbf{T}</math></p> <p>For <math>j =  \mathbf{m}  + 1, \dots, m : d_j = \mathbf{x}_j^\top \mathbf{t}; \mathbf{D}_j = \mathbf{x}_j^\top \mathbf{T}</math></p> <p>Return <math>([\mathbf{t}]_2, [u]_2, [\mathbf{T}]_2, [\mathbf{u}]_2, ([d_j]_2, [\mathbf{D}_j]_2)_{ \mathbf{m}  &lt; j \leq m})</math></p>
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**Figure 19:** Games  $G_0, (G_{1,i}, G'_{1,i})_{1 \leq i \leq Q}, G_{1,Q+1}, G_2$  for the proof of Theorem 5.9.

**Lemma A.11**  $\Pr[\text{APR-CMA}_{\text{real}}^A \Rightarrow 1] = \Pr[G_0^A \Rightarrow 1]$ .

In games  $G_{1,i}$ , for the first  $i - 1$  queries to the EVAL oracle,  $(\mathbf{t}, u, \mathbf{T}, \mathbf{u})$  is answered with uniformly random values and the rest are answered as in the real scheme. To interpolate between  $G_{1,i}$  and  $G_{1,i+1}$ , we also define  $G'_{1,i}$ , which answers the  $i$ -th query to EVAL by picking random  $\mathbf{t} \leftarrow_{\$} \mathbb{Z}_q^{k+1}$  and  $\mathbf{T} \leftarrow_{\$} \mathbb{Z}_q^{(k+1) \times k}$ . By the same arguments as in Appendix A.2, we have the following two lemmas:

**Lemma A.12**  $\Pr[G_0^A \Rightarrow 1] = \Pr[G'_{1,1}^A \Rightarrow 1]$ .

**Lemma A.13** *There exists an adversary  $\mathcal{B}$  with  $\mathbf{T}(\mathcal{B}) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}) \geq |\Pr[G'_{1,i}^A \Rightarrow 1] - \Pr[G_{1,i}^A \Rightarrow 1]|$ .*

**Lemma A.14**  $|\Pr[G'_{1,i+1}^A \Rightarrow 1] - \Pr[G'_{1,i}^A \Rightarrow 1]| \leq 1/q$ .

**Proof:** Similar to Lemma A.8, let  $\mathbf{m}$  be the  $i$ -th query to EVAL. As  $\mathbf{m} \neq \mathbf{m}^*$ , there exists an index  $i'$  such that  $\mathbf{m}_{i'} \neq \mathbf{m}_{i'}^*$ , where  $\mathbf{m}_{i'}$  (resp.  $\mathbf{m}_{i'}^*$ ) denotes the  $i'$ -th entry of  $\mathbf{m}$  (resp.  $\mathbf{m}^*$ ). We apply an information-theoretical argument to show in  $G'_{1,i}$  the values  $u$  and  $\mathbf{u}$  are uniformly random. For simplicity, we assume  $x'_0$  and  $\mathbf{x}_j$  ( $j \notin \{0, i'\}$ ) are learned by  $\mathcal{A}$ . Similar to the proof of Lemma 3.11, we assume  $\mathcal{A}$  learns  $\mathbf{B}^\top \mathbf{x}_0$  and  $\mathbf{B}^\top \mathbf{x}_{i'}$ . By the definition of APR-CMA security (Definition 5.8),  $\mathbf{m}$  can not be a prefix of  $\mathbf{m}^*$ . Thus, either  $i' \leq |\mathbf{m}^*|$  (Case 1) or  $i' > |\mathbf{m}^*| + 1$  and  $\mathbf{m}_j^* = m_j$  for all  $j \leq |\mathbf{m}^*|$  (Case 2).

**Case 1:** From the execution of  $G'_{1,i}$ ,  $\mathcal{A}$  information-theoretically obtains the following equations in the unknown variables  $\mathbf{x}_0, \mathbf{x}_{i'}$ :

$$\begin{pmatrix} \mathbf{B}^\top \mathbf{x}_0 \\ \mathbf{B}^\top \mathbf{x}_{i'} \\ \mathbf{h}_0 \\ u - x'_0 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{B}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^\top \\ h \cdot \mathbf{I}_{k+1} & \mathbf{m}_{i'}^* h \cdot \mathbf{I}_{k+1} \\ \mathbf{t}^\top & \mathbf{m}_{i'} \mathbf{t}^\top \\ \mathbf{T}^\top & \mathbf{m}_{i'} \mathbf{T}^\top \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_{i'} \end{pmatrix},$$

where  $\mathbf{h}_0$  is from  $\text{CHAL}(\mathbf{m}^*)$ ,  $u$  and  $\mathbf{u}$  are from  $\text{EVAL}(\mathbf{m})$ , and  $\mathbf{I}_{k+1}$  is the  $(k+1) \times (k+1)$  identity matrix. Since  $(\mathbf{t}, \mathbf{T})$  is chosen uniformly from  $\mathbb{Z}_q^{(k+1) \times (k+1)}$  in  $\mathbf{G}'_{1,i}$ ,  $\mathbf{t}$  and  $\mathbf{T}$  are not in the span of  $\mathbf{B}$  and  $(\mathbf{t}, \mathbf{T})$  has rank  $k+1$  (except with probability at most  $1/q$ ), which implies  $u - x'_0$  and  $\mathbf{u}$  are independent of  $\mathbf{B}^\top \mathbf{x}_0$  and  $\mathbf{B}^\top \mathbf{x}_{i'}$ . By  $\mathbf{m}_{i'} \neq \mathbf{m}_{i'}^*$ ,  $u - x'_0$  and  $\mathbf{u}$  are also independent of  $\mathbf{h}_0$ .

**Case 2:** Similarly, from the execution of  $\mathbf{G}'_{1,i}$ ,  $\mathcal{A}$  information-theoretically obtains the following equations in the unknown variables  $\mathbf{x}_0, \mathbf{x}_{i'}$ :

$$\begin{pmatrix} \mathbf{B}^\top \mathbf{x}_0 \\ \mathbf{B}^\top \mathbf{x}_{i'} \\ \mathbf{h}_0 \\ u - x'_0 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{B}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^\top \\ h \cdot \mathbf{I}_{k+1} & \mathbf{0} \\ \mathbf{t}^\top & \mathbf{m}_{i'} \mathbf{t}^\top \\ \mathbf{T}^\top & \mathbf{m}_{i'} \mathbf{T}^\top \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_{i'} \end{pmatrix}.$$

As in Case 1,  $u - x'_0$  and  $\mathbf{u}$  are linearly independent of  $\mathbf{B}^\top \mathbf{x}_0$  and  $\mathbf{B}^\top \mathbf{x}_i$ . It is easy to see that  $u - x'_0$  and  $\mathbf{u}$  are linearly independent of  $\mathbf{h}_0$ .

We conclude  $u$  and  $\mathbf{u}$  are distributed uniformly at random in  $\mathbf{G}'_{1,i}$ .  $\blacksquare$

**Lemma A.15**  $\Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}'_{1,Q+1} \Rightarrow 1]$ .

**Proof:** Note that  $\mathcal{A}$  can ask at most  $Q$ -many  $\text{EVAL}$  queries. In both  $\mathbf{G}_{1,Q+1}$  and  $\mathbf{G}_2$ , all answers of  $\text{EVAL}$  are uniformly random and independent of the secret keys  $(x'_0, \mathbf{x}_0, \dots, \mathbf{x}_L)$  where  $L = \max\{|\mathbf{m}_1|, \dots, |\mathbf{m}_Q|\}$ . Hence, the values  $\mathbf{h}_0$  and  $h_1$  from  $\mathbf{G}_{1,Q+1}$  are uniform in the view of  $\mathcal{A}$ .  $\blacksquare$

We apply the above arguments in a reverse order, we have the following lemma.

**Lemma A.16** *There exists an adversary  $\mathcal{B}$  with  $\mathbf{T}(\mathcal{B}) \approx \mathbf{T}(\mathcal{A})$  and  $|\Pr[\text{APR-CMA}_{\text{rand}}^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1]| \leq Q(\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}) + 1/q)$ .*

## B Security of the generic transformations

### B.1 Security of HIBKEM transformation (Theorem 5.7)

The proof of Theorem 5.7 is similar to the one of Theorem 4.3. We define the sequence of games  $\mathbf{G}_0$ - $\mathbf{G}_4$  as in Figure 20. Let  $\mathcal{A}$  be an adversary against the IND-HID-CPA security of  $\text{HIBKEM}[\text{MAC}, \mathcal{D}_k]$ .  $\mathbf{G}_0$  is the real attack game ( $\text{PR-HID-CPA}_{\text{real}}$ ) and  $\Pr[\mathbf{G}_0^{\mathcal{A}} \Rightarrow 1] = \Pr[\text{PR-HID-CPA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1]$ .

By applying the same arguments as in Lemma 4.4 and 4.5, one can easily show the following two lemmas.

**Lemma B.1**  $\Pr[\mathbf{G}_1^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_0^{\mathcal{A}} \Rightarrow 1]$ .

**Lemma B.2** *There exists an adversary  $\mathcal{B}_1$  with  $\mathbf{T}(\mathcal{B}_1) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}_1) \geq |\Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathbf{G}_1^{\mathcal{A}} \Rightarrow 1]|$ .*

**Lemma B.3**  $\Pr[\mathbf{G}_3^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1]$ .

<p><b>INITIALIZE:</b> // Games <math>G_0</math>-<math>G_2</math>, <math>\overline{G_3-G_4}</math></p> <p><math>\mathcal{G} \leftarrow_s \text{GGen}(1^\lambda)</math>; <math>\mathbf{A} \leftarrow_s \mathcal{D}_k</math></p> <p><math>\text{sk}_{\text{MAC}} = (\mathbf{B}, \mathbf{x}_0, \dots, \mathbf{x}_\ell, x'_0) \leftarrow_s \text{Gen}_{\text{MAC}}(\mathcal{G})</math></p> <p>For <math>i = 0, \dots, \ell</math>:</p> <p><math>\mathbf{Y}_i \leftarrow_s \mathbb{Z}_q^{k \times n}</math>; <math>\mathbf{Z}_i = (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \cdot \mathbf{A} \in \mathbb{Z}_q^{n \times k}</math></p> <p><math>\mathbf{d}_i = \mathbf{x}_i^\top \cdot \mathbf{B} \in \mathbb{Z}_q^{n'}</math>; <math>\mathbf{e}_i = \mathbf{Y}_i \cdot \mathbf{B} \in \mathbb{Z}_q^{k \times n'}</math></p> <p><math>\mathbf{E}_i = (\overline{\mathbf{A}}^{-1})^\top (\mathbf{Z}_i^\top \mathbf{B} - \mathbf{A}^\top \mathbf{d}_i)</math></p> <p><math>\mathbf{y}'_0 \leftarrow_s \mathbb{Z}_q^k</math>; <math>\mathbf{z}'_0 = (\mathbf{y}'_0^\top \mid x'_0) \cdot \mathbf{A} \in \mathbb{Z}_q^{1 \times k}</math></p> <p><math>\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, [\mathbf{z}'_0]_1)</math></p> <p><math>\text{dk} := ([\mathbf{B}]_2, ([\mathbf{d}_i]_2, [\mathbf{e}_i]_2)_{0 \leq i \leq \ell})</math></p> <p><math>\text{sk} := (\text{sk}_{\text{MAC}}, (\mathbf{Y}_i)_{0 \leq i \leq \ell}, \mathbf{y}'_0)</math></p> <p>Return <math>(\text{pk}, \text{dk})</math></p> <p><b>USKGEN(id):</b> // Games <math>G_0</math>-<math>G_2</math>, <math>\overline{G_3-G_4}</math></p> <p><math>\mathcal{Q}_{\text{ID}} = \mathcal{Q}_{\text{ID}} \cup \{\text{id}\}</math></p> <p><math>([\mathbf{t}]_2, [u]_2) \leftarrow_s \text{Tag}(\text{sk}_{\text{MAC}}, \text{id})</math></p> <p><math>\mathbf{v} = \sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Y}_i \mathbf{t} + \mathbf{y}'_0 \in \mathbb{Z}_q^k</math></p> <p><math>\mathbf{v}^\top = (\mathbf{t}^\top \sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Z}_i + \mathbf{z}'_0 - u \cdot \mathbf{A}) \cdot \overline{\mathbf{A}}^{-1}</math></p> <p>For <math>i = l(\text{id}) + 1, \dots, \ell</math>:</p> <p><math>d_i = \mathbf{x}_i^\top \mathbf{t} \in \mathbb{Z}_q</math></p> <p><math>\mathbf{e}_i = \mathbf{Y}_i \mathbf{t} \in \mathbb{Z}_q^k</math></p> <p><math>\mathbf{e}_i^\top = (\mathbf{t}^\top \mathbf{Z}_i - d_i \mathbf{A}) \overline{\mathbf{A}}^{-1} \in \mathbb{Z}_q^{1 \times k}</math></p> <p><math>\text{usk}[\text{id}] := ([\mathbf{t}]_2, [u]_2, [\mathbf{v}]_2) \in \mathbb{G}_2^n \times \mathbb{G}_2^1 \times \mathbb{G}_2^k</math></p> <p><math>\text{udk}[\text{id}] := ([d_i]_2, [\mathbf{e}_i]_2)_{l(\text{id}) &lt; i \leq \ell} \in (\mathbb{G}_2^{1+k})^{(\ell-l(\text{id}))}</math></p> <p>Return <math>(\text{usk}[\text{id}], \text{udk}[\text{id}])</math></p>	<p><b>ENC(id*):</b> // Games <math>G_0</math>, <math>\overline{G_1-G_2}</math>, <math>\overline{G_2}</math>, <math>G_3</math></p> <p><math>\mathbf{r} \leftarrow_s \mathbb{Z}_q^k</math></p> <p><math>\mathbf{c}_0^* = \mathbf{A} \mathbf{r} \in \mathbb{Z}_q^{k+1}</math></p> <p><math>[\mathbf{c}_0^*] \leftarrow_s \mathbb{Z}_q^{k+1}</math></p> <p><math>h \leftarrow_s \mathbb{Z}_q</math>; <math>\mathbf{c}_0^* \leftarrow_s \mathbb{Z}_q^k</math>; <math>\mathbf{c}_0^* := h + \mathbf{A} \cdot \overline{\mathbf{A}}^{-1} \mathbf{c}_0^* \in \mathbb{Z}_q^k</math></p> <p><math>\mathbf{c}_1^* = (\sum_{i=0}^{l(\text{id}^*)} f_i(\text{id}^*) \mathbf{Z}_i) \mathbf{r} \in \mathbb{Z}_q^n</math></p> <p><math>[\mathbf{c}_1^*] = \sum_{i=0}^{l(\text{id}^*)} f_i(\text{id}^*) (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \mathbf{c}_0^* \in \mathbb{Z}_q^n</math></p> <p><math>\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) (\mathbf{Z}_i \cdot \overline{\mathbf{A}}^{-1} \mathbf{c}_0^* + \mathbf{x}_i \cdot h)</math></p> <p><math>K^* = \mathbf{z}'_0 \cdot \mathbf{r} \in \mathbb{Z}_q</math></p> <p><math>K^* = (\mathbf{y}'_0^\top \mid x'_0) \mathbf{c}_0^* \in \mathbb{Z}_q</math></p> <p><math>K^* = \mathbf{z}'_0 \cdot \overline{\mathbf{A}}^{-1} \mathbf{c}_0^* + x'_0 \cdot h</math></p> <p>Return <math>K^* = [K^*]_T</math> and <math>C^* = ([\mathbf{c}_0^*]_1, [\mathbf{c}_1^*]_1)</math></p> <p><b>ENC(id*):</b> // Game <math>G_3</math>, <math>\overline{G_4}</math></p> <p><math>h \leftarrow_s \mathbb{Z}_q</math>; <math>\mathbf{c}_0^* \leftarrow_s \mathbb{Z}_q^k</math>; <math>\mathbf{c}_0^* := h + \mathbf{A} \cdot \overline{\mathbf{A}}^{-1} \mathbf{c}_0^* \in \mathbb{Z}_q^k</math></p> <p><math>\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) (\mathbf{Z}_i \cdot \overline{\mathbf{A}}^{-1} \mathbf{c}_0^* + \mathbf{x}_i \cdot h)</math></p> <p><math>K^* = \mathbf{z}'_0 \cdot \overline{\mathbf{A}}^{-1} \mathbf{c}_0^* + x'_0 \cdot h</math></p> <p><math>K^* \leftarrow_s \mathbb{Z}_q</math></p> <p>Return <math>K^* = [K^*]_T</math> and <math>C^* = ([\mathbf{c}_0^*]_1, [\mathbf{c}_1^*]_1)</math></p> <p><b>FINALIZE(<math>\beta</math>):</b> // Games <math>G_0</math>-<math>G_4</math></p> <p>Return <math>(\text{Prefix}(\text{id}^*) \cap \mathcal{Q}_{\text{ID}} = \emptyset) \wedge \beta</math></p>
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**Figure 20:** Games  $G_0$ - $G_4$  for the proof of IND-HID-CPA security (Theorem 5.7).

**Proof:** Similar to the proof of Lemma 4.6,  $G_3$  is simulated without using  $\mathbf{y}'_0$  and  $(\mathbf{Y}_i)_{0 \leq i \leq \ell}$ . By  $\mathbf{Y}_i^\top = (\mathbf{Z}_i - \mathbf{x}_i \mathbf{A}) \overline{\mathbf{A}}^{-1}$ , we have

$$\mathbf{E}_i = (\overline{\mathbf{A}}^{-1})^\top (\mathbf{Z}_i^\top \mathbf{B} - \mathbf{A}^\top \mathbf{d}_i) = \underbrace{(\overline{\mathbf{A}}^{-1})^\top (\mathbf{Z}_i^\top - \mathbf{A}^\top \mathbf{x}_i^\top)}_{\mathbf{Y}_i} \mathbf{B}$$

$$\mathbf{e}_i = (\overline{\mathbf{A}}^{-1})^\top \cdot (\mathbf{Z}_i^\top \mathbf{t} - \mathbf{A}^\top \underbrace{\mathbf{x}_i^\top \mathbf{t}}_{d_i}) = \mathbf{Y}_i \mathbf{t}.$$

as in Game  $G_2$ . By the same argument as Lemma 4.6, we have  $[\mathbf{v}]_2$ ,  $K^*$  and  $C^*$  are identical to  $G_2$ .  $\blacksquare$

**Lemma B.4** *There exists an adversary  $\mathcal{B}_2$  with  $\mathbf{T}(\mathcal{B}_2) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\text{MAC}}^{\text{hpr}_0\text{-cma}}(\mathcal{B}_2) \geq |\Pr[G_4^{\mathcal{A}} \Rightarrow 1] - \Pr[G_3^{\mathcal{A}} \Rightarrow 1]|$*

**Proof:** In  $G_4$ , we answer the  $\text{ENC}(\text{id}^*)$  query by choosing random  $K^*$ . We construct algorithm  $\mathcal{B}_2$  in Figure 21 to show the differences between  $G_4$  and  $G_3$  is bounded by the advantage of breaking  $\text{hpr}_0\text{-cma}$  security of MAC.

We note that, in games  $G_3$  and  $G_4$ , the values  $\mathbf{x}_i$  and  $x'_i$  are hidden until the call to  $\text{ENC}(\text{id}^*)$ . In both games  $\text{HPR-CMA}_{\text{real}}$  and  $\text{HPR}_0\text{-CMA}_{\text{rand}}$ , we have  $h = \mathbf{c}_0^* - \mathbf{A} \overline{\mathbf{A}}^{-1} \mathbf{c}_0^*$ . Hence  $\mathbf{h}_0 = \sum f_i(m_i) \mathbf{x}_i \cdot (\mathbf{c}_0^* - \mathbf{A} \cdot \overline{\mathbf{A}}^{-1} \mathbf{c}_0^*)$  which implies  $\mathbf{c}_1^*$  is distributed identically in games  $G_3$  and  $G_4$ . If  $h_1$  is uniform (i.e.,  $\mathcal{B}_2$  is in Game  $\text{HPR}_0\text{-CMA}_{\text{rand}}$ ) then the view of  $\mathcal{A}$  is the same as in  $G_4$ . If  $h_1$  is real (i.e.,  $\mathcal{B}_2$  is in Game  $\text{HPR-CMA}_{\text{real}}$ ) then  $K^* = \mathbf{z}'_0 \cdot \overline{\mathbf{A}}^{-1} \mathbf{c}_0^* + x'_0 \cdot h$ , which means the view of  $\mathcal{A}$  is the same as in  $G_3$ .  $\blacksquare$

The proof follows by combining Lemmas B.1-B.4 and observing that  $G_4 = \text{IND-HID-CPA}_{\text{rand}}$ .

<p><u>INITIALIZE:</u>  <math>\mathbf{A} \leftarrow_{\\$} \mathcal{D}_k</math>  <math>([\mathbf{B}]_2, ([\mathbf{x}_i^\top \mathbf{B}]_2)_{0 \leq i \leq \ell}) \leftarrow_{\\$} \text{INITIALIZE}_{\text{MAC}}</math>  For <math>i = 0, \dots, \ell</math>:  <math>\mathbf{Z}_i \leftarrow_{\\$} \mathbb{Z}_q^{n \times k}</math>; <math>\mathbf{d}_i := \mathbf{x}_i^\top \mathbf{B}</math>  <math>\mathbf{E}_i = (\mathbf{A}^{-1})^\top (\mathbf{Z}_i^\top \mathbf{B} - \mathbf{A}^\top \mathbf{d}_i)</math>  <math>\mathbf{z}'_0 \leftarrow_{\\$} \mathbb{Z}_q^{1 \times k}</math>  <math>\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, [\mathbf{z}'_0]_1)</math>  <math>\text{dk} := ([\mathbf{B}]_2, ([\mathbf{d}_i]_2, [\mathbf{E}_i]_2)_{0 \leq i \leq \ell})</math>  Return <math>(\text{pk}, \text{dk})</math></p> <p><u>ENC(id*):</u> // only one query  <math>([h]_1, [\mathbf{h}_0]_1, [h_1]_T) \leftarrow_{\\$} \text{CHAL}(\text{id}^*)</math>  <math>\mathbf{c}_0^* \leftarrow_{\\$} \mathbb{Z}_q^k</math>; <math>\mathbf{c}_0^* := h + \mathbf{A} \cdot \mathbf{A}^{-1} \mathbf{c}_0^* \in \mathbb{Z}_q</math>  <math>\mathbf{c}_1^* = \sum_{i=0}^{l(\text{id}^*)} f_i(\text{id}^*) \mathbf{Z}_i \cdot \mathbf{A}^{-1} \mathbf{c}_0^* + \mathbf{h}_0</math>  <math>K^* = \mathbf{z}'_0 \cdot \mathbf{A}^{-1} \mathbf{c}_0^* + h_1</math>  Return <math>K^* = [K^*]_T</math> and <math>C^* = ([\mathbf{c}_0^*]_1, [\mathbf{c}_1^*]_1)</math></p>	<p><u>USKGEN(id):</u>  <math>\overline{\mathcal{Q}_{\text{ID}}} = \mathcal{Q}_{\text{ID}} \cup \{\text{id}\}</math>  <math>([\mathbf{t}]_2, [u]_2, ([d_i]_2)_{0 \leq i \leq \ell}) \leftarrow_{\\$} \text{EVAL}(\text{id})</math>  <math>\mathbf{v}^\top = (\mathbf{t}^\top \sum f_i(\text{id}) \mathbf{Z}_i + \mathbf{z}'_0 - u \cdot \mathbf{A}) \cdot (\mathbf{A})^{-1}</math>  For <math>i = l(\text{id}) + 1, \dots, \ell</math>:  <math>\mathbf{e}_i^\top = (\mathbf{t}^\top \mathbf{Z}_i - d_i \mathbf{A}) \mathbf{A}^{-1} \in \mathbb{Z}_q^{1 \times k}</math>  <math>\text{usk}[\text{id}] := ([\mathbf{t}]_2, [u]_2, [\mathbf{v}]_2) \in \mathbb{G}_2^n \times \mathbb{G}_2^1 \times \mathbb{G}_2^k</math>  <math>\text{udk}[\text{id}] := ([d_i]_2, [\mathbf{e}_i]_2)_{l(\text{id}) &lt; i \leq \ell} \in (\mathbb{G}_2^{1+k})^{(\ell - l(\text{id}))}</math>  Return <math>(\text{usk}[\text{id}], \text{udk}[\text{id}])</math></p> <p><u>FINALIZE(<math>\beta</math>):</u>  Return <math>(\text{Prefix}(\text{id}^*) \cap \mathcal{Q}_{\text{ID}} = \emptyset) \wedge \text{FINALIZE}_{\text{MAC}}(\beta)</math></p>
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**Figure 21:** Description of  $\mathcal{B}_2$  (having access to the oracles  $\text{INITIALIZE}_{\text{MAC}}$ ,  $\text{EVAL}$ ,  $\text{CHAL}$ ,  $\text{FINALIZE}_{\text{MAC}}$  of the  $\text{HPR-CMA}_{\text{real}}/\text{HPR-CMA}_{\text{rand}}$  games of Figure 11) for the proof of Lemma B.4.

## B.2 Security of the Anonymous HIBKEM transformation (Theorem 5.10)

The proof of Theorem 5.10 is similar to the one of Theorem 5.7. We define the sequence of games  $\mathbf{G}_0$ - $\mathbf{G}_4$  in Figure 22. Let  $\mathcal{A}$  be an adversary against the PR-HID-CPA security of  $\text{AHIBKEM}[\text{MAC}, \mathcal{D}_k]$ .  $\mathbf{G}_0$  is the real attack game ( $\text{PR-HID-CPA}_{\text{real}}$ ) and  $\Pr[\mathbf{G}_0^{\mathcal{A}} \Rightarrow 1] = \Pr[\text{PR-HID-CPA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1]$ .

Analogously to Lemmas B.1 and B.2, we have

**Lemma B.5**  $\Pr[\mathbf{G}_1^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_0^{\mathcal{A}} \Rightarrow 1]$ .

**Lemma B.6** *There exists an adversary  $\mathcal{B}_1$  with  $\mathbf{T}(\mathcal{B}_1) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}_1) \geq |\Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathbf{G}_1^{\mathcal{A}} \Rightarrow 1]|$ .*

**Lemma B.7**  $\Pr[\mathbf{G}_3^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1]$ .

**Proof:**  $\mathbf{G}_3$  is defined without using  $\mathbf{y}'_0$  and  $(\mathbf{Y}_i)_{0 \leq i \leq \ell}$ . By Lemma B.3, we have values  $[\mathbf{v}]_2, [\mathbf{e}_i]_2, [\mathbf{E}_i]_2, K^*$  and  $C^*$  are identical in both  $\mathbf{G}_3$  and  $\mathbf{G}_2$ . By  $\mathbf{Y}_i^\top = (\mathbf{Z}_i - \mathbf{x}_i \mathbf{A}) \mathbf{A}^{-1}$ , we have

$$\begin{aligned} \mathbf{V} &= \sum f_i(\text{id}) (\mathbf{A}^{-1})^\top (\mathbf{Z}_i^\top - \mathbf{A}^\top \mathbf{x}_i^\top) \mathbf{T} = (\mathbf{A}^{-1})^\top \left( \sum f_i(\text{id}) \mathbf{Z}_i^\top \mathbf{T} - \mathbf{A}^\top \underbrace{\sum f_i(\text{id}) \mathbf{x}_i^\top \mathbf{T}}_{\mathbf{u}} \right) \\ \mathbf{E}_i &= (\mathbf{A}^{-1})^\top (\mathbf{Z}_i^\top - \mathbf{A}^\top \mathbf{x}_i^\top) \mathbf{T} = (\mathbf{A}^{-1})^\top (\mathbf{Z}_i^\top \mathbf{T} - \mathbf{A}^\top \underbrace{\mathbf{x}_i^\top \mathbf{T}}_{\mathbf{D}_i}). \end{aligned}$$

Thus,  $\mathbf{G}_3$  is identical to  $\mathbf{G}_2$ .  $\blacksquare$

**Lemma B.8** *There exists an adversary  $\mathcal{B}_2$  with  $\mathbf{T}(\mathcal{B}_2) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\text{MAC}}^{\text{apr-cma}}(\mathcal{B}_2) \geq |\Pr[\mathbf{G}_4^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathbf{G}_3^{\mathcal{A}} \Rightarrow 1]|$ .*

**Proof:** In  $\mathbf{G}_4$ , we answer the  $\text{ENC}(\text{id}^*)$  query by choosing random  $K^*$  and  $C^*$ . We construct algorithm  $\mathcal{B}_2$  in Figure 23 to show the differences between  $\mathbf{G}_4$  and  $\mathbf{G}_3$  is bounded by the advantage of breaking APR-CMA security of MAC.

We note that, in both  $\mathbf{G}_3$  and  $\mathbf{G}_4$ , the values  $\mathbf{x}_i$  and  $\mathbf{x}'_0$  are hidden until the call to  $\text{ENC}(\text{id}^*)$ . It is easy to see  $\mathbf{c}_0^*$  is uniform, since  $h$  and  $\bar{\mathbf{c}}_0^*$  are chosen uniformly at random. If  $(\mathbf{h}_0, h_1)$  is uniform (i.e.  $\mathcal{B}_2$  is

<p><b>INITIALIZE:</b> // Games <math>G_0</math>-<math>G_4</math></p> <p><math>\mathcal{G} \leftarrow_s \text{GGen}(1^\lambda); \mathbf{A} \leftarrow_s \mathcal{D}_k</math></p> <p><math>\text{sk}_{\text{MAC}} = (\mathbf{B}, \mathbf{x}_0, \dots, \mathbf{x}_\ell, x'_0) \leftarrow_s \text{Gen}_{\text{MAC}}(\mathcal{G})</math></p> <p>For <math>i = 0, \dots, \ell</math>:</p> <p><math>\mathbf{Y}_i \leftarrow_s \mathbb{Z}_q^{k \times n}; \mathbf{Z}_i = (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \cdot \mathbf{A} \in \mathbb{Z}_q^{n \times k}</math></p> <p><math>\mathbf{y}'_0 \leftarrow_s \mathbb{Z}_q^k; \mathbf{z}'_0 = (\mathbf{y}'_0^\top \mid x'_0) \cdot \mathbf{A} \in \mathbb{Z}_q^{1 \times k}</math></p> <p><math>\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, [\mathbf{z}'_0]_1)</math></p> <p><math>\text{sk} := (\text{sk}_{\text{MAC}}, (\mathbf{Y}_i)_{0 \leq i \leq \ell}, \mathbf{y}'_0)</math></p> <p>Return pk</p> <p><b>USKGEN(id):</b> // Games <math>G_0</math>-<math>G_2</math>, <math>G_3</math>-<math>G_4</math></p> <p><math>\mathcal{Q}_{\text{ID}} = \mathcal{Q}_{\text{ID}} \cup \{\text{id}\}</math></p> <p><math>([\mathbf{t}]_2, [u]_2) \leftarrow_s \text{Tag}(\text{sk}_{\text{MAC}}, \text{id})</math></p> <p><math>\mathbf{v} = \sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Y}_i \mathbf{t} + \mathbf{y}'_0 \in \mathbb{Z}_q^k</math></p> <p><math>\mathbf{v}^\top = (\mathbf{t}^\top \sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Z}_i + \mathbf{z}'_0 - u \cdot \mathbf{A}) \cdot \bar{\mathbf{A}}^{-1}</math></p> <p><math>\mathbf{S} \leftarrow_s \mathbb{Z}_q^{n' \times \mu}; \mathbf{T} = \mathbf{B} \cdot \mathbf{S} \in \mathbb{Z}_q^{n \times \mu};</math></p> <p><math>\mathbf{u} = \sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{x}_i^\top \mathbf{T} \in \mathbb{Z}_q^{1 \times \mu}</math></p> <p><math>\mathbf{V} = \sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Y}_i \mathbf{T} \in \mathbb{Z}_q^{k \times \mu}</math></p> <p><math>\mathbf{V} = (\bar{\mathbf{A}}^{-1})^\top (\sum_{i=0}^{l(\text{id})} f_i(\text{id}) \mathbf{Z}_i^\top \cdot \mathbf{T} - \mathbf{A}^\top \cdot \mathbf{u})</math></p> <p>For <math>i = l(\text{id}) + 1, \dots, \ell</math>:</p> <p><math>d_i = \mathbf{x}_i^\top \mathbf{t} \in \mathbb{Z}_q; \mathbf{D}_i = \mathbf{x}_i^\top \mathbf{T} \in \mathbb{Z}_q^{1 \times \mu}</math></p> <p><math>\mathbf{e}_i = \mathbf{Y}_i \mathbf{t} \in \mathbb{Z}_q^k; \mathbf{E}_i = \mathbf{Y}_i \mathbf{T} \in \mathbb{Z}_q^{k \times \mu}</math></p> <p><math>\mathbf{e}_i^\top = (\mathbf{t}^\top \mathbf{Z}_i - d_i \mathbf{A}) \bar{\mathbf{A}}^{-1} \in \mathbb{Z}_q^{1 \times k}</math></p> <p><math>\mathbf{E}_i = (\bar{\mathbf{A}}^{-1})^\top (\mathbf{Z}_i^\top \mathbf{T} - \mathbf{A}^\top \cdot \mathbf{D}_i) \in \mathbb{Z}_q^{k \times \mu}</math></p> <p><math>\text{usk}[\text{id}] := ([\mathbf{t}]_2, [u]_2, [\mathbf{V}]_2)</math></p> <p><math>\text{udk}[\text{id}] := ([\mathbf{T}]_2, [\mathbf{u}]_2, [\mathbf{V}]_2, ([d_i]_2, [\mathbf{D}_i]_2, [\mathbf{e}_i]_2, [\mathbf{E}_i]_2)_{l(\text{id}) &lt; i \leq \ell})</math></p> <p>Return <math>(\text{usk}[\text{id}], \text{udk}[\text{id}])</math></p>	<p><b>ENC(id*):</b> // Games <math>G_0</math>, <math>G_1</math>-<math>G_2</math>, <math>G_2</math>, <math>G_3</math></p> <p><math>\mathbf{r} \leftarrow_s \mathbb{Z}_q^k</math></p> <p><math>\mathbf{c}_0^* = \mathbf{A} \mathbf{r} \in \mathbb{Z}_q^{k+1}</math></p> <p><math>\mathbf{c}_0^* \leftarrow_s \mathbb{Z}_q^{k+1}</math></p> <p><math>h \leftarrow_s \mathbb{Z}_q; \mathbf{c}_0^* \leftarrow_s \mathbb{Z}_q^k; \mathbf{c}_0^* := h + \mathbf{A} \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* \in \mathbb{Z}_q</math></p> <p><math>\mathbf{c}_1^* = (\sum_{i=0}^{l(\text{id}^*)} f_i(\text{id}^*) \mathbf{Z}_i) \mathbf{r} \in \mathbb{Z}_q^n</math></p> <p><math>\mathbf{c}_1^* = \sum_{i=0}^{l(\text{id}^*)} f_i(\text{id}^*) (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \mathbf{c}_0^* \in \mathbb{Z}_q^n</math></p> <p><math>\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) (\mathbf{Z}_i \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* + \mathbf{x}_i \cdot h)</math></p> <p><math>K^* = \mathbf{z}'_0 \cdot \mathbf{r} \in \mathbb{Z}_q.</math></p> <p><math>K^* = (\mathbf{y}'_0^\top \mid x'_0) \mathbf{c}_0^* \in \mathbb{Z}_q</math></p> <p><math>K^* = \mathbf{z}'_0 \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* + x'_0 \cdot h</math></p> <p>Return <math>K^* = [K^*]_T</math> and <math>C^* = ([c_0^*]_1, [c_1^*]_1).</math></p> <p><b>ENC(id*):</b> // Game <math>G_4</math></p> <p><math>K^* \leftarrow_s \mathbb{G}_T; C^* \leftarrow_s \mathbb{G}_1^{n+k+1}</math></p> <p>Return <math>K^* = [K^*]_T</math> and <math>C^* = ([c_0^*]_1, [c_1^*]_1).</math></p> <p><b>FINALIZE(<math>\beta</math>):</b> // Games <math>G_0</math>-<math>G_4</math></p> <p>Return <math>(\text{Prefix}(\text{id}^*) \cap \mathcal{Q}_{\text{ID}} = \emptyset) \wedge \beta.</math></p>
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**Figure 22:** Games  $G_0$ - $G_4$  for the proof of PR-HID-CPA security (Theorem 5.10).

in Game  $\text{APR-CMA}_{\text{rand}}$ ) then the view of  $\mathcal{A}$  is the same as in  $G_4$ . If  $(\mathbf{h}_0, h_1)$  is real (i.e.  $\mathcal{B}_2$  is in Game  $\text{APR-CMA}_{\text{real}}$ ) then the view of  $\mathcal{A}$  is the same as in  $G_3$ . ■

The proof follows by combining Lemmas B.5-B.8 and observing that  $G_4 = \text{PR-HID-CPA}_{\text{rand}}$ .



<p><b>INITIALIZE:</b>  <math>\mathbf{A} \leftarrow_{\\$} \mathcal{D}_k; \varepsilon \leftarrow_{\\$} \text{INITIALIZE}_{\text{MAC}}</math>  For <math>i = 0, \dots, \ell</math>: <math>\mathbf{Z}_i \leftarrow_{\\$} \mathbb{Z}_q^{n \times k}</math>  <math>\mathbf{z}'_0 \leftarrow_{\\$} \mathbb{Z}_q^{1 \times k}</math>  <math>\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, [\mathbf{z}'_0]_1)</math>  Return <math>\text{pk}</math></p> <p><b>ENC(id*):</b> //only one query  <math>([h]_1, [\mathbf{h}_0]_1, [h_1]_T) \leftarrow_{\\$} \text{CHAL}(\text{id}^*)</math>  <math>\mathbf{c}_0^* \leftarrow_{\\$} \mathbb{Z}_q^k; \mathbf{c}_0^* := h + \mathbf{A} \cdot \mathbf{A}^{-1} \mathbf{c}_0^* \in \mathbb{Z}_q</math>  <math>\mathbf{c}_1^* = \sum_{i=0}^{\ell(\text{id}^*)} f_i(\text{id}^*) \mathbf{Z}_i \cdot \mathbf{A}^{-1} \mathbf{c}_0^* + \mathbf{h}_0</math>  <math>K^* = \mathbf{z}'_0 \cdot \mathbf{A}^{-1} \mathbf{c}_0^* + h_1</math>  Return <math>K^* = [K^*]_T</math> and <math>\mathbf{C}^* = ([\mathbf{c}_0^*]_1, [\mathbf{c}_1^*]_1)</math>.</p>	<p><b>USKGen(id):</b>  <math>\mathcal{Q}_{\text{ID}} = \mathcal{Q}_{\text{ID}} \cup \{\text{id}\}</math>  <math>([t]_2, [u]_2, [\mathbf{T}]_2, [\mathbf{u}]_2, ([d_i]_2, [\mathbf{D}_i]_2)_{(m) &lt; i \leq \ell}) \leftarrow_{\\$} \text{EVAL}(\text{id})</math>  <math>\mathbf{v}^\top = (\mathbf{t}^\top \sum f_i(\text{id}) \mathbf{Z}_i + \mathbf{z}'_0 - u \cdot \mathbf{A}) \cdot (\mathbf{A})^{-1}</math>  <math>\mathbf{V} = (\mathbf{A}^{-1})^\top (\sum f_i(\text{id}) \mathbf{Z}_i^\top \cdot \mathbf{T} - \mathbf{A}^\top \cdot \mathbf{u})</math>  For <math>i = \ell(\text{id}) + 1, \dots, \ell</math>:  <math>\mathbf{e}_i^\top = (\mathbf{t}^\top \mathbf{Z}_i - d_i \mathbf{A}) \mathbf{A}^{-1} \in \mathbb{Z}_q^{1 \times k}</math>  <math>\mathbf{E}_i = (\mathbf{A}^{-1})^\top (\mathbf{Z}_i^\top \mathbf{T} - \mathbf{A}^\top \cdot \mathbf{D}_i) \in \mathbb{Z}_q^{k \times \mu}</math>  <math>\text{usk}[\text{id}] := ([t]_2, [u]_2, [\mathbf{v}]_2)</math>  <math>\text{udk}[\text{id}] := ([\mathbf{T}]_2, [\mathbf{u}]_2, [\mathbf{V}]_2, ([d_i]_2, [\mathbf{D}_i]_2, [\mathbf{e}_i]_2, [\mathbf{E}_i]_2)_{\ell(\text{id}) &lt; i \leq \ell})</math>  Return <math>(\text{usk}[\text{id}], \text{udk}[\text{id}])</math>.</p> <p><b>FINALIZE(<math>\beta</math>):</b>  Return <math>(\text{Prefix}(\text{id}^*) \cap \mathcal{Q}_{\text{ID}} = \emptyset) \wedge \text{FINALIZE}_{\text{MAC}}(\beta)</math>.</p>
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**Figure 23:** Description of  $\mathcal{B}_2$  (having access to the oracles  $\text{INITIALIZE}_{\text{MAC}}, \text{EVAL}, \text{CHAL}, \text{FINALIZE}_{\text{MAC}}$  of the  $\text{APR-CMA}_{\text{real}}/\text{APR-CMA}_{\text{rand}}$  games of Figure 14) for the proof of Lemma B.8.

## C Identity-based Hash Proof System

In this section, we will show that our IBE construction from Section 4 gives a secure identity-based hash proof system (ID-HPS) [1], which implies IND-CCA secure [18] and leakage-resilient [1] IBE. Moreover, one of our constructions is the first tightly secure ID-HPS without a “ $q$ -type” assumption in prime-order groups.

### C.1 Definitions

We recall syntax and security of ID-HPS from [1].

**Definition C.1 (Identity-based Hash Proof System)** *An identity-based hash proof system (ID-HPS) consists of five PPT algorithms  $\text{IDHPS} = (\text{Setup}, \text{USKGen}, \text{Encap}, \text{Encap}^*, \text{Decap})$  with the following properties.*

- The probabilistic key generation algorithm  $\text{Setup}(\text{par})$  returns the (master) public/secret key  $(\text{pk}, \text{sk})$ . We assume that  $\text{pk}$  implicitly defines an identity space  $\mathcal{ID}$ , an encapsulated-key set  $\mathcal{K}$ .
- The probabilistic user secret key generation algorithm  $\text{USKGen}(\text{sk}, \text{id})$  returns the secret key  $\text{usk}[\text{id}]$  for an identity  $\text{id} \in \mathcal{ID}$ .
- The probabilistic valid encapsulation algorithm  $\text{Encap}(\text{pk}, \text{id})$  returns a pair  $(c, K)$  where  $c$  is a valid ciphertext, and  $K \in \mathcal{K}$  is the encapsulated-key with respect to identity  $\text{id}$ .
- The probabilistic invalid encapsulation algorithm  $\text{Encap}^*(\text{pk}, \text{id})$  samples an invalid ciphertext  $c$ .
- The deterministic decapsulation algorithm  $\text{Decap}(\text{usk}[\text{id}], c)$  returns a decapsulated key  $K$ .

For perfect correctness we require that for all  $\lambda \in \mathbb{N}$ , all pairs  $(\text{pk}, \text{sk})$  generated by  $\text{Setup}(1^\lambda)$ , all identities  $\text{id} \in \mathcal{ID}$ , all  $\text{usk}[\text{id}]$  generated by  $\text{USKGen}(\text{sk}, \text{id})$  and all  $(c, K)$  output by  $\text{Encap}(\text{pk}, \text{id})$ :

$$\Pr[\text{Decap}(\text{usk}[\text{id}], \text{id}, c) = K] = 1.$$

The security requirements for an ID-HPS are valid/invalid ciphertext indistinguishability (VI-IND) and smoothness. VI-IND security is defined via the games  $\text{VI-IND}_{\text{real}}$  and  $\text{VI-IND}_{\text{rand}}$  in Figure 24. Note that VI-IND security game, the adversary is allowed to ask for  $\text{usk}[\text{id}^*]$  (where  $\text{id}^*$  is the challenge identity) and that  $\text{USKGen}(\text{id})$  returns the same answer when queried twice on the same identity.

**Definition C.2 (VI-IND Security)** *An identity-based hash proof system IDHPS is VI-IND-secure if for all PPT  $\mathcal{A}$ ,  $\text{Adv}_{\text{IDHPS}}^{\text{vi-ind}}(\mathcal{A}) := |\Pr[\text{VI-IND}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] - \Pr[\text{VI-IND}_{\text{rand}}^{\mathcal{A}} \Rightarrow 1]|$  is negligible.*

Smoothness is a statistical property saying that the decapsulated key  $K$  for an *invalid* ciphertext is distributed statistically close to the uniform distribution.



<p><u>INITIALIZE:</u>  <math>\text{sk}_{\text{MAC}} \leftarrow_{\\$} \text{Gen}_{\text{MAC}}(\text{par})</math>  Return <math>\varepsilon</math></p> <p><u>CHAL(<math>\mathbf{m}^*</math>):</u> //one query  <math>h \leftarrow_{\\$} \mathbb{Z}_q</math>; <math>\mathbf{h}_0[\mathbf{m}^*] = \sum f_i(\mathbf{m}^*) \mathbf{x}_i \in \mathbb{Z}_q^n</math>  <math>([\mathbf{t}]_2, [\mathbf{u}]_2) \leftarrow_{\\$} \text{EVAL}(\mathbf{m}^*)</math>  Return <math>([h]_1, [\mathbf{h}_0[\mathbf{m}^*] \cdot h]_1)</math></p>	<p><u>EVAL(<math>\mathbf{m}</math>):</u>  If <math>(\mathbf{t}[\mathbf{m}], u[\mathbf{m}]) = (\perp, \perp)</math> then  <math>([\mathbf{t}[\mathbf{m}]]_2, [u[\mathbf{m}]]_2) \leftarrow_{\\$} \text{Tag}(\text{sk}_{\text{MAC}}, \mathbf{m})</math>  <math>[\mathbf{t}[\mathbf{m}]] \leftarrow_{\\$} \mathbb{Z}_q^n</math>; <math>\mathbf{h}_0[\mathbf{m}] \leftarrow_{\\$} \mathbb{Z}_q^n</math>; <math>h_1[\mathbf{m}] \leftarrow_{\\$} \mathbb{Z}_q</math>  <math>u[\mathbf{m}] = \mathbf{h}_0[\mathbf{m}] \cdot \mathbf{t}[\mathbf{m}] + h_1[\mathbf{m}]</math>  Return <math>([\mathbf{t}[\mathbf{m}]]_2, [u[\mathbf{m}]]_2)</math></p> <p><u>FINALIZE(<math>d \in \{0, 1\}</math>):</u>  Return <math>d</math></p>
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**Figure 26:** Games  $\text{SPR-CMA}_{\text{real}}$  and  $\text{SPR-CMA}_{\text{rand}}$  for defining SPR-CMA security.

### C.3 Security

We show that our ID-HPS is both smooth and VI-IND secure.

For the proof of VI-IND security, we require a security notion for the affine MAC that is slightly stronger than PR-CMA security in the sense that we allow the adversary to query EVAL with the challenge message  $\mathbf{m}^*$  in order to fit the definition of VI-IND security. We call it *strong pseudorandomness against chosen-message attacks* (SPR-CMA) defined via the security games in Figure 26.

**Definition C.5** An affine MAC over  $\mathbb{Z}_q^n$  is SPR-CMA-secure if for all PPT  $\mathcal{A}$ ,  $\text{Adv}_{\text{MAC}}^{\text{spr-cma}}(\mathcal{A}) := \Pr[\text{SPR-CMA}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] - \Pr[\text{SPR-CMA}_{\text{rand}}^{\mathcal{A}} \Rightarrow 1]$  is negligible, where the experiments are defined in Figure 26.

We note that both  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$  and  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  are SPR-CMA secure. The security proof from Section 3 can be easily adapted to show both schemes are SPR-CMA secure. Here we just outline the ideas. For  $\text{MAC}_{\text{NR}}[\mathcal{D}_k]$ , we first use the  $Q$ -fold  $\mathcal{D}_k$ -MDDH assumption to make the answers all EVAL queries random; next, we store a list of  $(\mathbf{h}_0[\cdot], h_1[\cdot])$  values to make the output of  $\text{CHAL}(\mathbf{m}^*)$  random and consistent with  $\text{EVAL}(\mathbf{m}^*)$ . One can also adapt the proof of  $\text{MAC}_{\text{HPS}}[\mathcal{D}_k]$  in a similar way.

The following theorem shows VI-IND security of IDHPS[MAC,  $\mathcal{D}_k$ ].

**Theorem C.6** Under the  $\mathcal{D}_k$ -MDDH assumption relative to GGen in  $\mathbb{G}_1$  and SPR-CMA security of MAC, IDSPHF[MAC,  $\mathcal{D}_k$ ] is a VI-IND secure ID-SPHF. Particularly, for all adversaries  $\mathcal{A}$  there exist adversaries  $\mathcal{B}_1$  and  $\mathcal{B}_2$  with  $\mathbf{T}(\mathcal{B}_1) \approx \mathbf{T}(\mathcal{A}) \approx \mathbf{T}(\mathcal{B}_2)$  and  $\text{Adv}_{\text{IDHPS}[\text{MAC}, \mathcal{D}_k]}^{\text{vi-ind}}(\mathcal{A}) \leq \text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}_1) + \text{Adv}_{\text{MAC}}^{\text{spr-cma}}(\mathcal{B}_2)$ .

**Proof:** The proof is similar to the one of Theorem 4.3 except that we need to simulate the user secret key for the challenge identity  $\text{id}^*$ . The games are defined as in Figure 27.

The following three lemmas and their proofs are exactly the same as Lemmas 4.4 to 4.6.

**Lemma C.7**  $\Pr[\text{VI-IND}_{\text{real}}^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_1^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_0^{\mathcal{A}} \Rightarrow 1]$ .

**Lemma C.8** There exists an adversary  $\mathcal{B}_1$  with  $\mathbf{T}(\mathcal{B}_1) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\mathcal{D}_k, \text{GGen}}(\mathcal{B}_1) \geq |\Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathbf{G}_1^{\mathcal{A}} \Rightarrow 1]|$ .

**Lemma C.9**  $\Pr[\mathbf{G}_3^{\mathcal{A}} \Rightarrow 1] = \Pr[\mathbf{G}_2^{\mathcal{A}} \Rightarrow 1]$ .

**Lemma C.10** There exists an adversary  $\mathcal{B}_2$  with  $\mathbf{T}(\mathcal{B}_2) \approx \mathbf{T}(\mathcal{A})$  and  $\text{Adv}_{\text{MAC}}^{\text{spr-cma}}(\mathcal{B}_2) \geq |\Pr[\mathbf{G}_4^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathbf{G}_3^{\mathcal{A}} \Rightarrow 1]|$ .

**Proof:** We construct an adversary  $\mathcal{B}_2$  in Figure 28 to show that the difference between  $\mathbf{G}_4$  and  $\mathbf{G}_3$  is bounded by the advantage of breaking SPR-CMA security of MAC.

By the definition of SPR-CMA, if  $\mathcal{B}_2$  is in Game  $\text{SPR-CMA}_{\text{rand}}$  then the view of  $\mathcal{A}$  is the same as in  $\mathbf{G}_4$ ; and if  $\mathcal{B}_2$  is in Game  $\text{SPR-CMA}_{\text{real}}$  then the view of  $\mathcal{A}$  is the same as in  $\mathbf{G}_3$ . ■

<p><u>INITIALIZE:</u> // Games <math>G_0</math>-<math>G_4</math></p> <p><math>\mathcal{G} \leftarrow_{\\$} \text{GGen}(1^\lambda); \mathbf{A} \leftarrow_{\\$} \mathcal{D}_k</math></p> <p><math>\text{sk}_{\text{MAC}} = (\mathbf{B}, \mathbf{x}_0, \dots, \mathbf{x}_\ell, x'_0, \dots, x'_{\ell'}) \leftarrow_{\\$} \text{Gen}_{\text{MAC}}(\mathcal{G})</math></p> <p>For <math>i = 0, \dots, \ell</math>: <math>\mathbf{Y}_i \leftarrow_{\\$} \mathbb{Z}_q^{k \times n}; \mathbf{Z}_i = (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \cdot \mathbf{A} \in \mathbb{Z}_q^{n \times k}</math></p> <p>For <math>i = 0, \dots, \ell'</math>: <math>\mathbf{y}'_i \leftarrow_{\\$} \mathbb{Z}_q^k; \mathbf{z}'_i = (\mathbf{y}'_i^\top \mid x'_i) \cdot \mathbf{A} \in \mathbb{Z}_q^{1 \times k}</math></p> <p><math>\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, ([\mathbf{z}'_i]_1)_{0 \leq i \leq \ell'})</math></p> <p><math>\text{sk} := (\text{sk}_{\text{MAC}}, (\mathbf{Y}_i)_{0 \leq i \leq \ell}, (\mathbf{y}'_i)_{0 \leq i \leq \ell'})</math></p> <p>Return pk</p> <p><u>USKGEN(id):</u> // Games <math>G_0</math>-<math>G_2</math>, <math>[G_3-G_4]</math>, <math>[G_4]</math></p> <p>If <math>\text{usk}[\text{id}] = \perp</math> then</p> <p><math>([\mathbf{t}]_2, [u]_2) \leftarrow_{\\$} \text{Tag}(\text{sk}_{\text{MAC}}, \text{id})</math></p> <p><math>\mathbf{t} \leftarrow_{\\$} \mathbb{Z}_q^n; \mathbf{h}_0[\text{id}] \leftarrow_{\\$} \mathbb{Z}_q^n; h_1[\text{id}] \leftarrow_{\\$} \mathbb{Z}_q^1</math></p> <p><math>[\mathbf{u}] = \mathbf{h}_0[\text{id}] \mathbf{t} + h_1[\text{id}]</math></p> <p><math>\mathbf{v} = \sum_{i=0}^{\ell} f_i(\text{id}) \mathbf{Y}_i \mathbf{t} + \sum_{i=0}^{\ell'} f'_i(\text{id}) \mathbf{y}'_i \in \mathbb{Z}_q^n</math></p> <p><math>[\mathbf{v}^\top] = (\mathbf{t}^\top \sum f_i(\text{id}) \mathbf{Z}_i + \sum f'_i(\text{id}) \mathbf{z}'_i - u \cdot \mathbf{A}) \cdot \bar{\mathbf{A}}^{-1}</math></p> <p><math>\text{usk}[\text{id}] := ([\mathbf{t}]_2, [u]_2, [\mathbf{v}]_2) \in \mathbb{G}_2^n \times \mathbb{G}_2^1 \times \mathbb{G}_2^k</math></p> <p>Return <math>\text{usk}[\text{id}]</math></p>	<p><u>ENC(id*):</u> // Games <math>G_0</math>, <math>[G_1-G_2]</math>, <math>[G_2]</math>, <math>G_3</math></p> <p><math>\mathbf{r} \leftarrow_{\\$} \mathbb{Z}_q^k</math></p> <p><math>\mathbf{c}_0^* = \mathbf{A} \mathbf{r} \in \mathbb{Z}_q^{k+1}; [\mathbf{c}_0^*] \leftarrow_{\\$} \mathbb{Z}_q^{k+1}</math></p> <p><math>h \leftarrow_{\\$} \mathbb{Z}_q; \mathbf{c}_0^* \leftarrow_{\\$} \mathbb{Z}_q^k; \mathbf{c}_0^* := h + \mathbf{A} \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* \in \mathbb{Z}_q</math></p> <p><math>\mathbf{c}_1^* = (\sum_{i=0}^{\ell} f_i(\text{id}^*) \mathbf{Z}_i) \mathbf{r} \in \mathbb{Z}_q^n</math></p> <p><math>[\mathbf{c}_1^*] = \sum_{i=0}^{\ell} f_i(\text{id}^*) (\mathbf{Y}_i^\top \mid \mathbf{x}_i) \mathbf{c}_0^* \in \mathbb{Z}_q^n</math></p> <p><math>\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) (\mathbf{Z}_i \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* + \mathbf{x}_i \cdot h)</math></p> <p>Return <math>\mathbf{C}^* = ([\mathbf{c}_0^*]_1, [\mathbf{c}_1^*]_1)</math></p> <p><u>ENC(id*):</u> // Game <math>G_4</math></p> <p>Call <math>\text{usk}[\text{id}^*] \leftarrow_{\\$} \text{USKGEN}(\text{id}^*)</math></p> <p><math>h \leftarrow_{\\$} \mathbb{Z}_q; \mathbf{c}_0^* \leftarrow_{\\$} \mathbb{Z}_q^k; \mathbf{c}_0^* := h + \mathbf{A} \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* \in \mathbb{Z}_q</math></p> <p><math>\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) (\mathbf{Z}_i \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^*) + \mathbf{h}_0[\text{id}^*] \cdot h</math></p> <p>Return <math>\mathbf{C}^* := ([\mathbf{c}_0^*]_1, [\mathbf{c}_1^*]_1)</math></p> <p><u>FINALIZE(<math>\beta'</math>):</u> // Game <math>G_1</math>-<math>G_4</math></p> <p>Return <math>\beta'</math></p>
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**Figure 27:** Games  $G_0$ - $G_4$  for the proof of the VI-IND security.

<p><u>INITIALIZE:</u></p> <p><math>\mathbf{A} \leftarrow_{\\$} \mathcal{D}_k</math></p> <p><math>\varepsilon \leftarrow_{\\$} \text{INITIALIZE}_{\text{MAC}}</math></p> <p>For <math>i = 0, \dots, \ell</math>: <math>\mathbf{Z}_i \leftarrow_{\\$} \mathbb{Z}_q^{n \times k}</math></p> <p>For <math>i = 0, \dots, \ell'</math>: <math>\mathbf{z}'_i \leftarrow_{\\$} \mathbb{Z}_q^{1 \times k}</math></p> <p><math>\text{pk} := (\mathcal{G}, [\mathbf{A}]_1, ([\mathbf{Z}_i]_1)_{0 \leq i \leq \ell}, ([\mathbf{z}'_i]_1)_{0 \leq i \leq \ell'})</math></p> <p>Return pk</p> <p><u>ENC(id*):</u> // one query</p> <p><math>([h]_1, [\mathbf{h}_0[\text{id}^*] \cdot h]_1) \leftarrow_{\\$} \text{CHAL}(\text{id}^*)</math></p> <p><math>\mathbf{c}_0^* \leftarrow_{\\$} \mathbb{Z}_q^k; \mathbf{c}_0^* = h + \mathbf{A} \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* \in \mathbb{Z}_q</math></p> <p><math>\mathbf{c}_1^* = \sum_{i=0}^{\ell} f_i(\text{id}^*) \mathbf{Z}_i \cdot \bar{\mathbf{A}}^{-1} \mathbf{c}_0^* + \mathbf{h}_0[\text{id}^*] \cdot h</math></p> <p>Return <math>\mathbf{C}^* = ([\mathbf{c}_0^*]_1, [\mathbf{c}_1^*]_1)</math></p>	<p><u>USKGEN(id):</u></p> <p>If <math>\text{usk}[\text{id}] = \perp</math> then</p> <p><math>([\mathbf{t}]_2, [u]_2) \leftarrow_{\\$} \text{EVAL}(\text{id})</math></p> <p><math>\mathbf{v}^\top = (\mathbf{t}^\top \sum f_i(\text{id}) \mathbf{Z}_i + \sum f'_i(\text{id}) \mathbf{z}'_i - u \cdot \mathbf{A}) \cdot \bar{\mathbf{A}}^{-1}</math></p> <p><math>\text{usk}[\text{id}] := ([\mathbf{t}]_2, [u]_2, [\mathbf{v}]_2) \in \mathbb{G}_2^n \times \mathbb{G}_2^1 \times \mathbb{G}_2^k</math></p> <p><math>\mathcal{Q}_{\text{ID}} = \mathcal{Q}_{\text{ID}} \cup \{(\text{id}, \text{usk}[\text{id}])\}</math></p> <p>Return <math>\text{usk}[\text{id}]</math></p>
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**Figure 28:** Description of  $\mathcal{B}_2$  (having access to the oracles  $\text{INITIALIZE}_{\text{MAC}}$ ,  $\text{EVAL}$ ,  $\text{CHAL}$ ,  $\text{FINALIZE}_{\text{MAC}}$  of the  $\text{PR-CMA}_{\text{real}}/\text{PR-CMA}_{\text{rand}}$  games of Figure 26) for the proof of Lemma C.10.

We observe that in Game  $G_4$   $\mathbf{c}_1^*$  is masked by the value  $\mathbf{h}_0[\text{id}^*]$ , which is uniformly random. The reason is that  $\mathbf{h}_0[\text{id}^*]$  is hidden from  $\text{USKGen}(\text{id}^*)$  query, since it is masked by a random  $h_1[\text{id}^*]$ . Thus,  $G_4 = \text{VI-IND}_{\text{rand}}$ . ■

**Theorem C.11**  $\text{IDHPS}[\text{MAC}, \mathcal{D}_k]$  is smooth.

**Proof:** We show that for almost all  $(\mathbf{c}_0, \mathbf{c}_1) \in \mathbb{Z}_q^{n+k+1}$ ,  $K = \mathbf{c}_0^\top \begin{pmatrix} \mathbf{v} \\ u \end{pmatrix} - \mathbf{c}_1^\top \mathbf{t}$  is uniformly random, where  $([t]_2, [u]_2, [v]_2) \leftarrow_{\$} \text{USKGen}(\text{id})$ . Similar to game  $G_3$  of VI-IND security proof, one can rewrite:

$$\begin{aligned} K &= \mathbf{c}_0^\top \begin{pmatrix} \mathbf{v} \\ u \end{pmatrix} - \mathbf{c}_1^\top \mathbf{t} \\ &= \mathbf{c}_0^\top \left( ((\mathbf{t}^\top \sum f_i(\text{id}) \mathbf{Z}_i + \sum_u f'_i(\text{id}) \mathbf{z}'_i - u \cdot \mathbf{A}) \cdot \bar{\mathbf{A}}^{-1})^\top \right) - \mathbf{c}_1^\top \mathbf{t} \\ &= \bar{\mathbf{c}}_0^\top (\bar{\mathbf{A}}^{-1})^\top \sum f'_i(\text{id}) \mathbf{z}'_i + (\mathbf{c}_0^\top - (\mathbf{A} \bar{\mathbf{A}}^{-1} \bar{\mathbf{c}}_0)^\top) u + ((\sum f_i(\text{id}) \mathbf{Z}_i \cdot \bar{\mathbf{A}}^{-1} \cdot \bar{\mathbf{c}}_0)^\top - \mathbf{c}_1^\top) \mathbf{t} \\ &= \bar{\mathbf{c}}_0^\top (\bar{\mathbf{A}}^{-1})^\top \sum f'_i(\text{id}) \mathbf{z}'_i + (\mathbf{c}_0^\top - (\mathbf{A} \bar{\mathbf{A}}^{-1} \bar{\mathbf{c}}_0)^\top) \sum f'_i(\text{id}) \mathbf{x}'_i \quad (\text{substituting } u) \\ &\quad + \underbrace{((\mathbf{c}_0^\top - (\mathbf{A} \bar{\mathbf{A}}^{-1} \bar{\mathbf{c}}_0)^\top) \sum f_i(\text{id}) \mathbf{x}_i^\top + (\sum f_i(\text{id}) \mathbf{Z}_i \cdot \bar{\mathbf{A}}^{-1} \cdot \bar{\mathbf{c}}_0)^\top - \mathbf{c}_1^\top) \mathbf{t}}_E \end{aligned}$$

Since  $(\mathbf{c}_0, \mathbf{c}_1)$  from  $\text{Encap}^*$  is chosen uniformly at random,  $\mathbf{c}_1^\top \neq (\mathbf{c}_0^\top - (\mathbf{A} \bar{\mathbf{A}}^{-1} \bar{\mathbf{c}}_0)^\top) \sum f_i(\text{id}) \mathbf{x}_i^\top + (\sum f_i(\text{id}) \mathbf{Z}_i \cdot \bar{\mathbf{A}}^{-1} \cdot \bar{\mathbf{c}}_0)^\top$  with probability  $1 - 1/q^n$ . Conditioned on that and since  $\text{rank}(\mathbf{B}) \geq 1$  and  $\mathbf{t} = \mathbf{B}\mathbf{s}$  (for  $\mathbf{s} \leftarrow_{\$} \mathbb{Z}_q^{n'}$ ),  $E$  is uniformly random. This concludes the theorem. ■

## D Concrete Instantiation from SXDH

We now describe our tightly PR-ID-CPA-secure IBKEM  $\text{IBKEM}[\text{MAC}_{\text{NR}}[\mathcal{D}_k], \mathcal{U}_1]$  for the special case of  $k = 1$  and  $\mathcal{D}_k = \mathcal{U}_1$ , such that  $\mathcal{D}_k\text{-MDDH}$  is the DDH assumption. The identity space is  $\mathcal{ID} = \{0, 1\}^\ell$ . Theorems 4.3 and 3.3 provide a tight security reduction under the DDH assumptions in  $\mathbb{G}_1$  and  $\mathbb{G}_2$ , i.e., under the SXDH assumption.

<p><b>Gen(par):</b>  <math>\text{sk}_{\text{MAC}} = (x_0, \dots, x_\ell, x') \leftarrow_{\\$} \mathbb{Z}_q^{\ell+2}</math>  <math>a \leftarrow_{\\$} \mathbb{Z}_q</math>  For <math>i = 0, \dots, \ell</math>: <math>y_i \leftarrow_{\\$} \mathbb{Z}_q</math>; <math>z_i = ay_i + x_i \in \mathbb{Z}_q</math>  <math>y' \leftarrow_{\\$} \mathbb{Z}_q</math>; <math>z' = ay' + x' \in \mathbb{Z}_q</math>  <math>\text{pk} := (\mathcal{G}, [a]_1, ([z_i]_1)_{0 \leq i \leq \ell}, [z']_1)</math>  <math>\text{sk} := (\text{sk}_{\text{MAC}}, (y_i)_{0 \leq i \leq \ell}, y')</math>  Return <math>(\text{pk}, \text{sk})</math>.</p> <p><b>USKGen(sk, id):</b>  <math>t \leftarrow_{\\$} \mathbb{Z}_q</math>  <math>u = x' + t(x_0 + \sum_{i=1}^\ell \text{id}_i \cdot x_i)</math>  <math>v = y' + t(y_0 + \sum_{i=1}^\ell \text{id}_i \cdot y_i)</math>  Return <math>\text{usk}[\text{id}] := ([t]_2, [u]_2, [v]_2) \in \mathbb{G}_2^3</math></p>	<p><b>Enc(pk, id):</b>  <math>r \leftarrow_{\\$} \mathbb{Z}_q</math>  <math>\mathbf{c}_0 := (c_{0,0}, c_{0,1}) = (r, a \cdot r)</math>  <math>\mathbf{c}_1 = r(z_0 + \sum_{i=1}^\ell \text{id}_i z_i)</math>  <math>K = z' \cdot r</math>  Return <math>\mathbf{K} = [K]_T</math> and <math>\mathbf{C} = ([c_0]_1, [c_1]_1) \in \mathbb{G}_1^2</math>.</p> <p><b>Dec(usk[id], id, C):</b>  Parse <math>\text{usk}[\text{id}] = ([t]_2, [u]_2, [v]_2)</math>  Parse <math>\mathbf{C} = ([c_0]_1, [c_1]_1)</math>  <math>\mathbf{K} = e([c_{0,0}]_1, [v]_2) \cdot e([c_{0,1}]_1, [u]_2) / e([c_1]_1, [t]_2)</math>  Return <math>\mathbf{K} \in \mathbb{G}_T</math>.</p>
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$\text{IBKEM}[\text{MAC}_{\text{NR}}[\mathcal{D}_k], \mathcal{U}_1]$  is a ‘‘Cramer-Shoup variant’’ of Waters’ IBKEM Wat05 [28]. Concretely, Wat05 is a projected variant of our scheme and is obtained by setting  $\text{usk}[\text{id}] := ([t]_2, [u + av]_2) \in \mathbb{G}_2^2$  and  $\mathbf{C} = ([c_0 = c_{0,0} + ac_{0,1}], [c_1]) \in \mathbb{G}_1^2$ . Wat05 is IND-ID-CPA-secure under the CDH assumption, with a non-tight security proofs. See [3] for a discussion on the impact of the non-tight reduction. Our IBKEM is tightly IND-ID-CPA-secure and anonymous under the SXDH assumption.