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Certification of Minimal Approximant Bases

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ABSTRACT
For a given computational problem, a certificate is a piece of data that one (the prover) attaches to the output with the aim of allowing efficient verification (by the verifier) that this output is correct. Here, we consider the minimal approximant basis problem, for which the fastest known algorithms output a polynomial matrix of dimensions $m \times m$ and average degree $D/m$ using $O(m^d D/m)$ field operations. We propose a certificate which, for typical instances of the problem, is computed by the prover using $O(m^{d^2} D/m)$ additional field operations and allows verification of the approximant basis by a Monte Carlo algorithm with cost bound $O(m^d + mD)$.

Besides theoretical interest, our motivation also comes from the fact that approximant bases arise in most of the fastest known algorithms for linear algebra over the univariate polynomials; thus, this work may help in designing certificates for other polynomial matrix computations. Furthermore, cryptographic challenges such as breaking records for discrete logarithm computations or for integer factorization rely in particular on computing minimal approximant bases for large instances: certificates can then be used to provide reliable computation on outsourced and error-prone clusters.

KEYWORDS
Certification; minimal approximant basis; order basis; polynomial matrix; truncated product.

1 INTRODUCTION
Context. For a given tuple $d = (d_1, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n$ called order, we consider an $m \times n$ matrix $F$ of formal power series with the column $j$ truncated at order $d_j$. Formally, let $F \in \mathbb{K}[X]^{m \times n}$ be a matrix over the univariate polynomials over a field $\mathbb{K}$, such that the column $j$ of $F$ has degree less than $d_j$. Then, we consider the classical notion of minimal approximant bases for $F$ [1, 27]. An approximant is a polynomial row vector $p \in \mathbb{K}[X]^{1 \times m}$ such that

$$pF \equiv 0 \mod X^d, \quad \text{where } X^d = \text{diag}(x^{d_1}, \ldots, x^{d_n}); \quad (1)$$

here $pF \equiv 0 \mod X^d$ means that $pF = qX^d$ for some $q \in \mathbb{K}[X]^{1 \times n}$. The set of all approximants forms a (free) $\mathbb{K}[X]$-module of rank $m$,

$$\mathcal{A}_d(F) = \left\{ p \in \mathbb{K}[X]^{1 \times m} \mid pF \equiv 0 \mod X^d \right\}.$$
rather than $C = AB$. This verification algorithm costs $O(n^3)$ and
is false-biased one-sided Monte-Carlo (it is always correct when it
answers "false"); the probability of error can be made arbitrarily
small by picking several random vectors.

In some cases, one may require an additional piece of data to be
produced together with the output in order to prove the correctness
of the result. For example, Farkas’ lemma [10] certifies the infeasi-
bility of a linear program thanks to an extra vector. Although the
verification is deterministic in this example, the design of certifi-
cates that are verified by probabilistic algorithms opened a line of
work for faster certification methods in linear algebra [7, 8, 17, 18].

In this context, one of the main challenges is to design optimal
certificates, that is, ones which are verifiable in linear time. Further-
more, the time and space needed for the certificate must remain
negligible. In this work, we seek such an optimal certificate for the
problem of computing shifted minimal approximate bases.

Here, an instance is given by the input $(d, F, s)$ which is of size
$O(mD)$: each column $j$ of $F$ contains at most $md_j$ elements of $\mathbb{X}$,
and the order sums to $d_1 + \cdots + d_n = |d| = D$. We neglect the size of
the shift $s$, since one may always assume that it is nonnegative and
such that $\max(s) < mD$ (see [15, App. A]). Thus, ideally one would
like to have a certificate which can be verified in time $O(mD)$.

In this paper, we provide a non-interactive certification protocol
which uses the input $(d, F, s)$, the output $P$, and a certificate which is
a constant matrix $C \in \mathbb{K}^{m \times n}$. We design a Monte-Carlo verification
algorithm with cost bound $O(mD + m^2 n + n)$; this is optimal
as soon as $D$ is large compared to $m$ and $n$ (e.g. when $D > m^2 + mn$),
which is most often of the interest. We also show that the
certificate $C$ can be computed in $O(m^{o(1)}D)$ operations in $\mathbb{X}$, which
is faster than known approximate basis algorithms.

**Degrees and size of approximate bases.** For $P \in \mathbb{X}^{m \times m}$, we
denote the row degree of $P$ as $\deg_i(P) = (r_1, \ldots, r_m)$ where $r_i = \deg(P_{i, \cdot})$ is the degree of the row $i$ of $P$ for $1 \leq i \leq m$. The column
degree $\deg_j(P)$ is defined similarly. More generally, we will consider
row degrees shifted by some additive column weights: for a shift $s =
(s_1, \ldots, s_m) \in \mathbb{Z}^m$ the $s$-row degree of $P$ is $\deg_{s, i}(P) = (r_1 + s_1, \ldots, r_m + s_m)$.

We use $\| \cdot \|$ to denote the sum of integer tuples: for example
$\|\deg_j(P)\|$ is the sum of the $s$-row degree of $P$ (note that this sum
might contain negative terms). The comparison of integer tuples is
entrywise: $\deg(F) < d$ means that the column $j$ of $P$ has degree
less than $d_j$, for $1 \leq j \leq n$. When adding a constant to a tuple, say
for example $s - 1$, this stands for the tuple $(s_1 - 1, \ldots, s_m - 1)$.

In existing approximate basis algorithms, the output bases may
take different forms: essentially, they can be s-minimal (also called
s-reduced [27]), s-weak Popov [23], or s-Popov [3]. For formal
definitions and for motivating the use of shifts, we direct the reader
to these references and to those above about approximate basis
algorithms; here the precise form of the basis will not play an
important role. What is however at the core of the efficiency of our
algorithms is the impact of these forms on the degrees in the basis.

In what follows, by size of a matrix we mean the number of field
elements used for its dense representation. We define the quantity

\[
\text{Size}(P) = m^2 + \sum_{1 \leq i, j \leq m} \max(0, \deg(p_{ij}))
\]

for a matrix $P = [p_{ij}] \in \mathbb{X}^{m \times m}$. In the next paragraph, we
discuss degree bounds on $P$ when it is the output of any of the
approximant basis algorithms mentioned above; note that these
bounds all imply that $P$ has size in $O(mD)$.

There is no general degree bound for approximate bases: any
unimodular matrix is a basis of $\mathcal{A}_d(0) = \mathbb{X}^{m \times m}$. Still, a basis $P$
of $\mathcal{A}_d(F)$ always satisfies $\deg(\det(P)) \leq D$. Now, for an s-minimal
$P$, we have $\deg(P) \in O(D)$ as soon as $|s - \min(s)| \in O(D)$ [27,
Thm. 4.1], and it was shown in [28] that $P$ has size in $O(mD)$ if
$\max(s) - |s| \in O(D)$. Yet, without such assumptions on the shift,
there are s-minimal bases whose size is in $\Theta(m^2 D)$ [15, App. B],
ruling out the feasibility of finding them in time $O(m^{o(1)}D)$. In this
case, the fastest known algorithms return the more constrained
s-Popov basis $P$, for which $\deg(P) \leq D$ holds independently of $s$.

**Problem and contribution.** Certifying that a matrix $P$ is an s-
minimal approximate basis for a given instance $(d, F, s)$ boils down
to the following three properties of $P$:

1. **Minimal**: $P$ is in s-reduced form. By definition, this amounts
to testing the invertibility of the so-called s-leading matrix
of $P$ (see Step 1 of Algorithm 1 for the construction of this
matrix), which can be done using $O(m^{o(1)})$ operations in $\mathbb{X}$.

2. **Approximant**: the rows of $P$ are approximants. That is, we
should check that $FP = 0 \mod X^d$. The difficulty is to avoid
computing the full truncated product $PF \mod X^d$, since this
costs $O(m^{o(1)}D)$. In Section 3, we give a probabilistic algo-
rithm which verifies more generally $PF = 0 \mod X^d$ using
$O(\text{Size}(P) + mD)$ operations, without requiring a certificate.

3. **Basis**: the rows of $P$ generate the approximant module. For
this, we prove that it suffices to verify first that $\det(P)$ is of the
form $cX^d$ for some $c \in \mathbb{X} \setminus \{0\}$ and where $\delta = \|\deg(P)\|$, and
second that some constant $m \times (m + n)$ matrix has full rank;
this matrix involves $P(0)$ and the coefficient $C$ of degree 0 of
$\text{PF}^{-d}$. In Section 2, we show that $C$ can serve as a certificate,
and that a probabilistic algorithm can assess its correctness
at a suitable cost.

Our (non-interactive) certification protocol is as follows. Given
$(d, F, s)$, the Prover computes a matrix $P$, supposedly an s-minimal
basis of $\mathcal{A}_d(F)$, along with a constant matrix $C \in \mathbb{K}^{m \times n}$, supposedly
the coefficient of degree 0 of the product $\text{PF}^{-d}$. Then, the Prover
communicates these results to the Verifier who must solve Problem 1
within a cost asymptotically better than $O(m^{o(1)}D)$.

**Problem 1: Approximant Basis Certification**

**Input:**

- order $d \in \mathbb{Z}_{\geq 0}$
- matrix $F \in \mathbb{X}^{m \times n}$ with $\deg(F) < d$
- shift $s \in \mathbb{Z}^m$
- matrix $P \in \mathbb{X}^{m \times m}$
- certificate matrix $C \in \mathbb{K}^{m \times n}$

**Output:**

- True if $P$ is an s-minimal basis of $\mathcal{A}_d(F)$ and $C$ is the
  coefficient of degree 0 of $\text{PF}^{-d}$, otherwise False.

---

1This is not implied by (1) and (2): for $d = \max(d)$, then $X^d I_m$ is s-reduced
and $X^d I_m F = 0 \mod X^d$ holds; yet, $X^d I_m$ is not a basis of $\mathcal{A}_d(F)$ for most $(F, d)$. 
The main result in this paper is an efficient solution to Problem 1.

**Theorem 1.1.** There is a Monte-Carlo algorithm which solves Problem 1 using $O(mD + m^{\omega-1}(m + n))$ operations in $\mathbb{K}$, assuming $\text{Size}(F) \in O(mD)$. It chooses $m + 2$ elements uniformly and independently at random from a finite subset $S \subset \mathbb{K}$. If $S$ has cardinality at least $2(D + 1)$, then the probability that a true answer is incorrect is less than $1/2$, while a false answer is always correct.

A detailed cost bound showing the constant factors is described in Proposition 2.5. If $\text{Size}(P) \in O(mD)$, then the size of the input of Problem 1 is in $O(mD)$; the cost bound above is therefore optimal (up to constant factors) as soon as $m^{\omega-2}(m + n) \in O(D)$.

If $\mathbb{K}$ is a small finite field, there may be no subset $S \subset \mathbb{K}$ of cardinality $|S| \geq 2(D + 1)$. Then, our approach still works by performing the probabilistic part of the computation over a sufficiently large extension of $\mathbb{K}$. Note that an extension of degree about $1 + \lceil \log_d(D) \rceil$ would be suitable; this would increase our complexity estimates by (up to constant factors) as soon as $m^{\omega-2}(m + n) \in O(D)$.

Let $d \in \mathbb{Z}^n_0$, let $F \in \mathbb{K}[X]^{m \times n}$ with $\deg(F) < d$ and $m \in O(D)$, and let $P \in \mathbb{K}[X]^{m \times m}$. If $\deg(P) \in O(D)$ or $\deg(P) \in O(D)$, there is a deterministic algorithm which computes the coefficient of degree $0$ of $\text{PFX}^{-d}$ using $O(m^{\omega-1}D)$ operations in $\mathbb{K}$ if $m \geq n$ and $O(m^{\omega-1}D)\log(n/m)$ operations in $\mathbb{K}$ if $m < n$.

Note that the assumption $m \in O(D)$ in this theorem is commonly made in approximant basis algorithms, since when $D \leq m$ most entries of a minimal approximant basis have degree in $O(1)$ and the algorithms then rely on methods from dense $\mathbb{K}$-linear algebra.

## 2 Certifying Approximant Bases

Here, we present our certification algorithm. Its properties, given in Proposition 2.5, prove Theorem 1.1. One of its core components is the verification of truncated polynomial matrix products; the details of this are in Section 3 and are taken for granted here.

First, we show the basic properties behind the correctness of this algorithm, which are summarized in the following result.

**Theorem 2.1.** Let $d \in \mathbb{Z}^n_0$, let $F \in \mathbb{K}[X]^{m \times n}$, and let $s \in \mathbb{Z}^n$. A matrix $P \in \mathbb{K}[X]^{m \times m}$ is an $s$-minimal basis of $\mathcal{A}_d(F)$ if and only if the following properties are all satisfied:

- (i) $P$ is $s$-reduced;
- (ii) $\det(P)$ is a nonzero monomial in $\mathbb{K}[X]$;
- (iii) the rows of $P$ are in $\mathcal{A}_d(F)$, that is, $PF = 0 \bmod X^d$;
- (iv) $[P(0) \ C] \in \mathbb{K}^{(m+n) \times m}$ has full rank, where $C$ is the coefficient of degree $0$ of $\text{PFX}^{-d}$.

We remark that having both $PF = 0 \bmod X^d$ and $C$ the constant coefficient of $\text{PFX}^{-d}$ is equivalent to the single truncated identity $PF = CX^d \bmod X^t$, where $t = (d_1 + 1, \ldots, d_n + 1)$.

As mentioned above, the details of the certification of the latter identity is deferred to Section 3, where we present more generally the certification for truncated products of the form $PF = G \bmod X^t$.

Concerning Item (ii), the fact that the determinant of any basis of $\mathcal{A}_d(F)$ must divide $X^D$, where $D = |d|$, is well-known; we refer to [2, Sec. 2] for a more general result.

The combination of Items (i) and (iii) describes the set of matrices $P \in \mathbb{K}[X]^{m \times m}$ which are $s$-reduced and whose rows are in $\mathcal{A}_d(F)$.
thus $U(0)$ has rank $< m$. Since $[P \| Q] = U[A \ AXF^{-d}]$, it directly follows that $[P(0) \| C]$ has rank $< m$. 

**Proof of Theorem 2.1.** If $P$ is an $s$-minimal basis of $\mathcal{A}_d(F)$, then by definition Items (i) and (iii) are satisfied. Since the rows of $X^{\max(d)}1_m$ are in $\mathcal{A}_d(F)$ and $P$ is a basis, the matrix $X^{\max(d)}1_m$ is a left multiple of $P$ and therefore the determinant of $P$ divides $X^{\max(d)}$: it is a nonzero monomial. Then, according to Lemma 2.4, $[P(0) \| C]$ has full rank. Conversely, if Items (ii) to (iv) are satisfied, then Lemma 2.4 states that $P$ is a basis of $\mathcal{A}_d(F)$; thus if furthermore Item (i) is satisfied then $P$ is an $s$-minimal basis of $\mathcal{A}_d(F)$. 

**Algorithm 1: CertifApproxBasis**

**Input:**
- order $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{> 0}$
- matrix $F \in \mathbb{R}[X]^{m \times n}$ with $cdeg(F) < d$
- shift $s = (s_1, \ldots, s_n) \in \mathbb{Z}^m$
- matrix $P \in \mathbb{R}[X]^{m \times m}$
- certificate matrix $C \in \mathbb{R}^{m \times n}$

**Output:** True if $P$ is an $s$-minimal basis of $\mathcal{A}_d(F)$ and $C$ is the constant term of $PFX^{-d}$; otherwise True or False.

1. /* $P$ not in $s$-reduced form */
   
   $L \leftarrow$ the matrix in $\mathbb{R}^{m \times n}$ whose entry $i, j$ is the coefficient of degree $rdeg_{P_{i,j}} - s_j$ of the entry $i, j$ of $P$
   
   If $L$ is not invertible then return False

2. /* rank($P(0) \| C)$ not full rank */
   
   $\text{rank}(P(0) \| C) < m$ then return False

3. /* det($P$) not a nonzero monomial */
   
   $S \leftarrow$ a finite subset of $\mathbb{R}$
   
   $\Delta \leftarrow |rdeg_{P(s)}| - |s|$  
   
   $a \leftarrow$ chosen uniformly at random from $S$
   
   If $\text{det}(P(a)) \neq \text{det}(P(1))a^\Delta$ then return False

4. /* certify truncated product $PF = CX^d \mod X$ */
   
   $t \leftarrow (d_1 + 1, \ldots, d_n + 1)$
   
   Return $\text{VerifTruncMatProd}(t, P, F, CX^d)$

In order to provide a sharp estimate of the cost of Algorithm 1, we recall the best known cost bound with constant factors of the LQUP factorization of an $m \times n$ matrix over $\mathbb{R}$, which we use for computing ranks and determinants. Assuming $m \leq n$, we have:

$$C(m, n) = \left( \left\lceil \frac{n}{m} \right\rceil \frac{1}{2^{n+1}} \frac{1}{2 \omega - 2} \right) MM(m)$$

operations in $\mathbb{R}$ [6, Lem. 5.1], where $MM(m)$ is the cost for the multiplication of $m \times m$ matrices over $\mathbb{R}$.

**Proposition 2.5.** Algorithm 1 uses at most

$S\text{Size}(P) + 2m(D + \max(d)) + 3C(m, m) + C(m, m + n)$

$+ (4m + n) \log_2(Dd_1 \cdots d_n)$

operations in $\mathbb{R}$, where $D = \lceil d \rceil$. It is a false-biased Monte Carlo algorithm. If $P$ is not an $s$-minimal basis of $\mathcal{A}_d(F)$, then the probability that it outputs True is less than $\frac{D_{s+1}}{2^{n+1}}$, where $S$ is the finite subset of $\mathbb{R}$ from which random field elements are drawn.

**Proof.** By definition, $P$ is $s$-reduced if and only if its $s$-leading matrix $L$ computed at Step 1 is invertible. Thus, Step 1 correctly tests the property in Item (i) of Theorem 2.1. It uses at most $C(m, m)$ operations in $\mathbb{R}$. Furthermore, Step 2 correctly tests the first part of Item (iv) of Theorem 2.1 and uses at most $C(m, m + n)$ operations.

Step 3 performs a false-biased Monte Carlo verification of Item (ii) of Theorem 2.1. Indeed, since $P$ is $s$-reduced (otherwise the algorithm would have exited at Step 1), we know from [16, Sec. 6.3.2] that $\text{deg}(\text{det}(P)) = \Delta = |rdeg_{P(s)}| - |s|$. Thus, $\text{det}(P)$ is a nonzero monomial if and only if $\text{det}(P) = \text{det}(P(1))a^\Delta$. Step 3 tests the latter equality by evaluation at a random point $a$. The algorithm only returns False if $\text{det}(P(a)) \neq \text{det}(P(1))a^\Delta$, in which case $\text{det}(P)$ is indeed not a nonzero monomial. Furthermore, if we have $\text{det}(P) \neq \text{det}(P(1))a^\Delta$, then the probability that the algorithm fails to detect this, meaning that $\text{det}(P(a)) = \text{det}(P(1))a^\Delta$, is at most $\frac{\Delta}{2^\Delta}$. Since $\Delta \leq D$ according to [27, Thm. 4.1], this is also at most $D \leq \frac{D+1}{2^\Delta}$. The evaluations $P(a)$ and $P(1)$ are computed using respectively at most $2\text{Size}(P) - m^2$ operations and at most $\text{Size}(P) - m^2$ additions. Then, computing the two determinants $\text{det}(P(a))$ and $\text{det}(P(1))$ uses at most $2C(m, m) + 2m$ operations. Finally, computing $\text{det}(P(1))a^\Delta$ uses at most $2\log_2(\Delta) + 1 < 2\log_2(D) + 1$ operations.

Summing the cost bounds for the first three steps gives

$3\text{Size}(P) - m^2 + 3C(m, m) + C(m, m + n) + 2m + 2\log_2(D) + 1 
\leq 3\text{Size}(P) + 3C(m, m) + C(m, m + n) + 2\log_2(D).$ (2)

Step 4 tests the identity $PF = CX^d \mod X^t$, which corresponds to both Item (iii) of Theorem 2.1 and the second part of Item (iv). Proposition 3.2 ensures that:

- If the call to $\text{VerifTruncMatProd}$ returns False, we have $PF \neq CX^d \mod X^t$, and Algorithm 1 correctly returns False.
- If $PF \neq CX^d \mod X^t$ holds, the probability that Algorithm 1 fails to detect this (that is, the call at Step 4 returns True) is less than $\frac{\text{max}(D+1, D^t)}{2^{n+1}}$.

A cost bound for Step 4 is given in Proposition 3.2, with a minor improvement for the present case given in Remark 3.3. Summing it with the bound in Eq. (2) gives a cost bound for Algorithm 1, which is bounded from above by that in the proposition.

Thanks to Theorem 2.1, the above considerations show that when the algorithm returns False, then $P$ is indeed not an $s$-minimal basis of $\mathcal{A}_d(F)$. On the other hand, if $P$ is not an $s$-minimal basis of $\mathcal{A}_d(F)$, the algorithm returns True if and only if one of the probabilistic verifications in Steps 3 and 4 take the wrong decision. According to the probabilities given above, this may happen with probability less than $\frac{\text{max}(D+1, D^t)}{2^{n+1}} = \frac{D_{s+1}}{2^{n+1}}$. 

**3 VERIFYING A TRUNCATED PRODUCT**

In this section, we focus on the verification of truncated products of polynomial matrices, and we give the corresponding algorithm $\text{VerifTruncMatProd}$ used in Algorithm 1.

Given a truncation order $t$ and polynomial matrices $P, F, G$, our goal is to verify that $PF = G \mod X^t$ holds with good probability. Without loss of generality, we assume that the columns of $F$ and $G$ are already truncated with respect to the order $t$, that is, $cdeg(F) < t$ and $cdeg(G) < t$. Similarly, we assume that $P$ is truncated with respect to $\delta = \max(t)$, that is, $\text{deg}(P) < \delta$. 


Problem 2: TRUNCATED MATRIX PRODUCT VERIFICATION

Input:
- truncation order \( t \in \mathbb{Z}_0^n \)
- matrix \( P \in \mathbb{K}[X]_{m \times n} \) with deg(\( P \)) < max(\( t \))
- matrix \( F \in \mathbb{K}[X]_{m \times n} \) with cdeg(\( F \)) < \( t \)
- matrix \( G \in \mathbb{K}[X]_{m \times n} \) with cdeg(\( G \)) < \( t \)

Output:
- True if \( PF = G \mod X^t \), otherwise False.

Obviously, our aim is to obtain a verification algorithm which has a significantly better cost than the straightforward approach which computes the truncated product \( PF \mod X^t \) and compares it with the matrix \( G \). To take an example: if we have \( n \in O(m) \) as well as \( \{\text{deg}(P)\} \in O(\|t\|) \) or \( \{\text{deg}(P)\} \in O(\|t\|) \), as commonly happens in approximant basis computations, then this truncated product \( PF \mod X^t \) can be computed using \( O(m\|t\|-1) \) operations in \( \mathbb{K} \).

For verifying the non-truncated product \( PF = G \), the classical approach would be to use evaluation at a random point, following ideas from [5, 25, 32]. However, evaluation does not behave well with regards to truncation. A similar issue was tackled in [13] with the truncated product \( uA \) the probability that \( \alpha \) is written \( \delta \)-biased, and as we detail now.

To lower down the cost, we will combine the evaluation of truncated products with matrix multiplication by writing them as univariate polynomials with matrix coefficients; for example, \( P \) is a polynomial \( P = \sum_{0 \leq i < \delta} P_i X^i \) with coefficients \( P_i \in \mathbb{K}[m \times n] \). While this leads to a verification of \( PF = G \mod X^t \) with a good probability of success, it has a cost which is close to that of computing \( PF \mod X^t \).

In what follows, given a matrix \( A \in \mathbb{K}[X]_{m \times n} \) and an order \( t \in \mathbb{Z}_0^n \), we write \( A \) rem \( X^t \) for the (unique) matrix \( B \) in \( \mathbb{K}[X]_{m \times n} \) such that \( B = A \mod X^t \) and cdeg(\( B \)) < \( t \). For simplicity, we will often write \( A_{1:2} \) to actually mean \( (A_1, A_2) \) rem \( X^t \).

Lemma 3.1. Let \( S \) be a finite subset of \( \mathbb{K} \). Let \( u \in \mathbb{K}^{1 \times m} \) with entries chosen uniformly and independently at random from \( S \), and let \( a \in \mathbb{K} \) be chosen uniformly at random from \( S \). Assuming \( PF \neq G \mod X^t \), the probability that \( \text{uPF \ rem \ X}^t(a) = uG(a) \) is less than \( \frac{\max(t)}{\delta^s} \).

Proof. Let \( A = (PF - G) \) rem \( X^t \). By assumption, there exists a pair \( (i, j) \) such that the entry \( (i, j) \) of \( A \) is nonzero. Since this entry is a polynomial in \( \mathbb{K}[X] \) of degree less than \( \max(t) \), the probability that \( \alpha \) is a root of this entry is at most \( \frac{\alpha}{\delta^s} \). As a consequence, we have \( A(\alpha) \neq 0 \in \mathbb{K}^{m \times n} \) with probability at least \( 1 - \frac{\delta^s}{\alpha} \). In this case, \( uA(\alpha) = 0 \) occurs with probability at most \( \frac{1}{\alpha} \) (see [22, Sec. 7.1]).

Thus, altogether the probability that \( uA(\alpha) = 0 \) is bounded from above by \( \frac{\alpha}{\delta^s} + \left(1 - \frac{\alpha}{\delta^s}\right) \frac{1}{\alpha} < \frac{\delta^s}{\alpha^2} \), which concludes the proof.

We deduce an approach to verify the truncated product: compute \( uA(a) = (\text{uPF} - uG) \text{rem} X^t(a) \) and check whether it is zero or nonzero. The remaining difficulty is to compute \( uA(a) \) efficiently; we will see that this can be done in \( O(\text{Size}(P) + m\|t\|) \) operations.

For this, we use a strategy similar to that in [15, Lem. 4.1] and essentially based on the following formula for the truncated product

\[
\text{Problem 2: TRUNCATED MATRIX PRODUCT VERIFICATION}
\]

\[
\text{Input:}
\]
- truncation order \( t \in \mathbb{Z}_0^n \)
- matrix \( P \in \mathbb{K}[X]_{m \times n} \) with deg(\( P \)) < max(\( t \))
- matrix \( F \in \mathbb{K}[X]_{m \times n} \) with cdeg(\( F \)) < \( t \)
- matrix \( G \in \mathbb{K}[X]_{m \times n} \) with cdeg(\( G \)) < \( t \)

\[
\text{Output:}
\]
- True if \( PF = G \mod X^t \), otherwise False.

Proof. Let \( A = (PF - G) \mod X^t \). By assumption, there exists a pair \( (i, j) \) such that the entry \( (i, j) \) of \( A \) is nonzero. Since this entry is a polynomial in \( \mathbb{K}[X] \) of degree less than \( \max(t) \), the probability that \( \alpha \) is a root of this entry is at most \( \frac{\alpha}{\delta^s} \). As a consequence, we have \( A(\alpha) \neq 0 \in \mathbb{K}^{m \times n} \) with probability at least \( 1 - \frac{\delta^s}{\alpha} \). In this case, \( uA(\alpha) = 0 \) occurs with probability at most \( \frac{1}{\alpha} \) (see [22, Sec. 7.1]).

Thus, altogether the probability that \( uA(\alpha) = 0 \) is bounded from above by \( \frac{\alpha}{\delta^s} + \left(1 - \frac{\alpha}{\delta^s}\right) \frac{1}{\alpha} < \frac{\delta^s}{\alpha^2} \), which concludes the proof.

We deduce an approach to verify the truncated product: compute \( uA(a) = (\text{uPF} - uG) \text{rem} X^t(a) \) and check whether it is zero or nonzero. The remaining difficulty is to compute \( uA(a) \) efficiently; we will see that this can be done in \( O(\text{Size}(P) + m\|t\|) \) operations.

For this, we use a strategy similar to that in [15, Lem. 4.1] and essentially based on the following formula for the truncated product.

Consider a positive integer \( t \leq \delta \) and a vector \( f \in \mathbb{K}[X]_{m \times 1} \) of degree less than \( t \); one may think of \( f \) as a column \( F_{s,j} \) of \( F \) and of \( t \) as the corresponding order \( t_j \). Writing \( f = \sum_{0 \leq k < \delta} f_k X^k \) with \( f_k \in \mathbb{K}^{\delta \times 1} \) and \( uP = \sum_{0 \leq k < \delta} p_k X^k \) with \( p_k \in \mathbb{K}^{\delta \times m} \), we have

\[
\text{Problem 2: TRUNCATED MATRIX PRODUCT VERIFICATION}
\]

\[
\text{Input:}
\]
- truncation order \( t \in \mathbb{Z}_0^n \)
- matrix \( P \in \mathbb{K}[X]_{m \times n} \) with deg(\( P \)) < max(\( t \))
- matrix \( F \in \mathbb{K}[X]_{m \times n} \) with cdeg(\( F \)) < \( t \)
- matrix \( G \in \mathbb{K}[X]_{m \times n} \) with cdeg(\( G \)) < \( t \)

\[
\text{Output:}
\]
- True if \( PF = G \mod X^t \), otherwise False.

Proof. Let \( A = (PF - G) \mod X^t \). By assumption, there exists a pair \( (i, j) \) such that the entry \( (i, j) \) of \( A \) is nonzero. Since this entry is a polynomial in \( \mathbb{K}[X] \) of degree less than \( \max(t) \), the probability that \( \alpha \) is a root of this entry is at most \( \frac{\alpha}{\delta^s} \). As a consequence, we have \( A(\alpha) \neq 0 \in \mathbb{K}^{m \times n} \) with probability at least \( 1 - \frac{\delta^s}{\alpha} \). In this case, \( uA(\alpha) = 0 \) occurs with probability at most \( \frac{1}{\alpha} \) (see [22, Sec. 7.1]).

Thus, altogether the probability that \( uA(\alpha) = 0 \) is bounded from above by \( \frac{\alpha}{\delta^s} + \left(1 - \frac{\alpha}{\delta^s}\right) \frac{1}{\alpha} < \frac{\delta^s}{\alpha^2} \), which concludes the proof.

We deduce an approach to verify the truncated product: compute \( uA(a) = (\text{uPF} - uG) \text{rem} X^t(a) \) and check whether it is zero or
Algorithm 2: VerifTruncMultProd
Input:
- truncation order \( t = (t_1, \ldots, t_n) \in \mathbb{Z}_+^n \)
- matrix \( P \in \mathbb{K}[X]^{m \times m} \) such that \( \deg(P) < \delta = \max(t) \),
- matrix \( F = [f_{ij}] \in \mathbb{K}[X]^{m \times n} \) with \( \deg(F) < t \)
- matrix \( G = [g_{ij}] \in \mathbb{K}[X]^{n \times n} \) with \( \deg(G) < t \)
Output: True if \( PF = G \bmod X^\delta \), otherwise True or False.
1. /* Main objects for verification */
   \( S \leftarrow \) a finite subset of \( \mathbb{K} \)
   \( a \leftarrow \) element of \( \mathbb{K} \) chosen uniformly at random from \( S \)
   \( u \leftarrow \) vector in \( \mathbb{K}^{m \times n} \) with entries chosen uniformly and independently at random from \( S \)
2. /* Freivalds: row dimension becomes 1 */
   \( p \leftarrow u^T P \) // in \( \mathbb{K}[X]^{1 \times m} \), degree \( < \delta \)
   \( g \leftarrow uG \) // in \( \mathbb{K}[X]^{1 \times n} \), \( \deg(g) < t \)
3. /* Evaluation of right-hand side: \( uPF \) */
   \( g = [g_1, \ldots, g_n] \) with \( g_j \in \mathbb{K}[X] \) of degree \( < t \)
   For \( j \) from 1 to \( n \):
      \( e_j \leftarrow g_j(a) \)
4. /* Truncated evaluations \( c_0, \ldots, c_{\delta-1} \) */
   \( p = \sum_{0 \leq k < \delta} p_k X^k \) with \( p_k \in \mathbb{K}^{1 \times m} \)
   \( c_0 \leftarrow p_0 \)
   For \( k \) from 1 to \( \delta - 1 \):
      \( c_k \leftarrow \sum_{l \leq k-t} p_l X^{k-l} \)
5. /* Evaluation of left-hand side: \( uPF \) */
   For \( j \) from 1 to \( n \):
      // process column \( F_{\cdot,j} \)
      \( \lambda_k \leftarrow \sum_{0 \leq k < t_j} f_{kj} X^k \)
      \( e_{t_j}^\prime \leftarrow \sum_{0 \leq k < t_j} \lambda_k \)
      \( e_j^\prime \leftarrow (\lambda_k)^{-1} \lambda_k \)
6. If \( e_j \neq e_j^\prime \) for some \( j \in \{1, \ldots, n\} \) then return False
   Else return True

\[ 2m|t| + 2 \log_2((t_1 - 1) \cdots (t_n - 1)) \] operations for Step 5. Finally, Step 6 uses at most \( n \) comparisons of two field elements. Summing these bounds for each step yields the cost bound

\[ 2\text{Size}(P) + (4m + 1)|t| + 2m(\delta - 1) - n + 2 \log_2((t_1 - 1) \cdots (t_n - 1)) \]

which is at most the quantity in the proposition. \( \square \)

In the certification of approximant bases, we want to verify a truncated matrix product in the specific case where each entry in the column \( j \) of \( G \) is simply zero or a monomial of degree \( t_j - 1 \). Then, a slightly better cost bound can be found, as follows.

Remark 3.3. Assume that \( t = (d_1, \ldots, d_n + 1) \) and \( G = CX^d \), for some \( d = (d_1, \ldots, d_n) \in \mathbb{Z}_+^n \) and some constant \( C \in \mathbb{K}^{m \times n} \). Then, the computation of \( uG \) at Step 2 uses at most \( (2m - 1)n \) operations in \( \mathbb{K} \). Besides, since the polynomial \( g_j \) at Step 3 is either zero or a monomial of degree \( d_j \), its evaluation \( e_j \) is computed using at most \( 2 \log_2(d_j) + 1 \) operations via repeated squaring [[12, Sec. 4.3]. Thus, Step 3 uses at most \( 2 \log_2(d_1 \cdots d_n) + n \) operations. As a result, defining \( D = |d| \), the cost bound in Eq. (5) is lowered to

\[ 2\text{Size}(P) + 2|t| + 2m(\delta - 1) + n + 4 \log_2(d_1 \cdots d_n) + 1 \]

\[ = 2\text{Size}(P) + 2m(D + \max(d)) + 2n + n + 4 \log_2(d_1 \cdots d_n) + 1 \]

\( \square \)

4 COMPUTING THE CERTIFICATE

4.1 Context

In this section, we show how to efficiently compute the certificate \( C \in \mathbb{K}^{m \times n} \), which is the term of degree \( 0 \) of the product \( PFX^{-d} \), whose entries are Laurent polynomials (they are in \( \mathbb{K}[X] \) if and only if the rows of \( P \) are approximants). Equivalently, the column \( C_{i,j} \) is the term of degree \( d_j \) of the column \( j \) of \( PF \), where \( d = (d_1, \ldots, d_n) \).

We recall the notation \( D = d_1 + \cdots + d_n \). Note that, without loss of generality, we may truncate \( P \) so that \( \deg(P) \leq \max(d) \).

For example, suppose that the dimensions and the order are balanced: \( m = n \) and \( d = (D/m, \ldots, D/m) \). Then, \( C \in \mathbb{K}^{m \times m} \) is the coefficient of degree \( D/m \) of the product \( PF \), where \( P \) and \( F \) are \( m \times m \) matrices over \( \mathbb{K} \). Thus \( C \) can be computed using \( D/m \) multiplications of \( m \times m \) matrices over \( \mathbb{K} \), at a total cost \( O(m^{n-1}D) \).

Going back to the general case, the main obstacle to obtain similar efficiency is that both the degrees in \( P \) and the order \( d \) (hence the degrees in \( F \)) may be unbalanced. Still, we have \( \deg(F) \leq d \) with sum \( |d| = D \) and, as stated in the introduction, we may assume that either \( |\deg(P)| \in O(D) \) or \( |\deg(P)| \leq D \) holds. In this context, both \( P \) and \( F \) are represented by \( O(mD) \) field elements.

We will generalize the method above for the balanced case to this general situation with unbalanced degrees, achieving the same cost \( O(m^{n-1}D) \). As a result, computing the certificate \( C \) has negligible cost compared to the fastest known approximate basis algorithms. Indeed, the latter are in \( O(m^{n-1}D) \), involving logarithmic factors in \( D \) coming both from polynomial arithmetic and from divide and conquer approaches. We refer the reader to [28, Thm. 5.3] and [15, Thm. 1.4] for more details on these logarithmic factors.

We first remark that \( C \) can be computed by naive linear algebra using \( O(m^2D) \) operations. Indeed, writing \( \deg(P) = (r_1, \ldots, r_m) \), we have the following explicit formula for each entry in \( C \):

\[ C_{i,j} = \sum_{k=1}^{\min(r_i,d_j)} P_{i,*k} F_{*,j,d_j-k} \]

where \( P_{i,*k} \) is the coefficient of degree \( k \) of the row \( i \) of \( P \) and similar notation is used for \( F \). Then, since \( \min(r_i,d_j) \leq d_j \), the column \( C_{i,j} \) is computed via \( m d_j \) scalar products of length \( m \), using \( O(m^2 d_j) \) operations. Summing this for \( 1 \leq j \leq n \) yields \( O(m^2 D) \).

This approach considers each column of \( F \) separately, allowing us to truncate at precision \( d_j + 1 \) for the column \( j \) and thus to rule out the issue of the unbalancedness of the degrees in \( P \). However, this also prevents us from incorporating fast matrix multiplication.

In our efficient method, we avoid considering columns or rows separately, while still managing to handle the unbalancedness of the degrees in both \( P \) and \( F \). Our approach bears similarities with algorithms for polynomial matrix multiplication with unbalanced degrees (see for example [31, Sec. 3.6]).
Since cdeg(F) < d, we have deg(H) < d. The fact that F has column degree less than d translates into the fact that H has column valuation at least d − d (and degree less than d); like F, this matrix H is represented by mD field elements. Recalling the assumption deg(P) ≤ d, we can write $P = \sum_{k=0}^{d} P_k X^k$ and $H = \sum_{k=0}^{d} H_k X^k$, where $P_k \in \mathbb{K}^{m \times m}$ and $H_k \in \mathbb{K}^{m \times d}$ for all k (note that $H_d = 0$).

Then, our goal is to compute the matrix

$$C = \sum_{k=1}^{d} P_k H_{d-k}.$$  \hspace{1cm} (6)

The essential remark to design an efficient algorithm is that each matrix $P_k$ has only few nonzero rows when k becomes large, and each matrix $H_{d-k}$ has only few nonzero columns when k becomes large. To state this formally, we define two sets of indices, for the rows of degree at least k in P and for the orders at least k in d:

$$\mathcal{R}_k = \{i \in \{1, \ldots, m\} \mid \text{rdeg}(P_{i,k}) \geq k\},$$

$$\mathcal{D}_k = \{j \in \{1, \ldots, n\} \mid d_j \geq k\}.$$  

The latter corresponds to the set of indices of columns of F which are allowed to have degree ≥ k − 1 or, equivalently, to the set of indices of columns of H which are allowed to have valuation ≤ d − k.

Proof. The row i of $P_k$ is the coefficient of degree k of the row i of P. If it is nonzero, we must have i ∈ $\mathcal{R}_k$. Similarly, the column j of $H_{d-k}$ is the coefficient of degree d − k of the column j of H = FX$^{-d}$. If it is nonzero, we must have $d - k \geq d - j_k$, hence k ∈ $\mathcal{D}_k$.

The upper bounds on the cardinalities of $\mathcal{R}_k$ and $\mathcal{D}_k$ follow by construction of these sets: we have $k \cdot |\mathcal{R}_k| \leq |\text{rdeg}(P)|$ with |rdeg(P)| ≤ yD by assumption. \hfill \Box

4.3 Algorithm and cost bound

Following Lemma 4.1. in the computation of C based on Eq. (6) we may restrict our view of $P_k$ to its submatrix with rows in $\mathcal{R}_k$, and our view of $H_k$ to its submatrix with columns in $\mathcal{D}_k$. For example, if $k > yD/m$ and $k > D/n$, the matrices in the product $P_k H_k$ have dimensions at most $[yD/k] \times m$ and $m \times [D/k]$. These remarks on the structure and sparsity of $P_k$ and $H_k$ lead us to Algorithm 3.2.

Proposition 4.2. Algorithm 3 is correct. Assuming that m ∈ O(D) and |rdeg(P)| ∈ O(D), where D = |d|, it uses O(m$^{e-1}$D) operations in $\mathbb{K}$ if n ≤ m and O(m$^{e-1}$D log(n/m)) operations in $\mathbb{K}$ if n > m.

Proof. For the correctness, note that for all j the coefficient of degree $d_j - k$ of $F_{s,i}$ is the coefficient of degree $d - k$ of $H_{s,j}$. Thus, using notation from Section 4.2, the matrix B at the iteration k of the loop is exactly the submatrix of $H_{d-k}$ of its columns in $\mathcal{D}_k$. Therefore, the loop in Algorithm 3 simply applies Eq. (6), discarding from $P_k$ and $F_{s,i}$ rows and columns which are known to be zero.

Now, we estimate the cost of updating C at each iteration of the loop. Precisely, the main task is to compute AB, where the matrices A and B have dimensions $|\mathcal{R}| \times m$ and $m \times t$. Then, adding this product to the submatrix $C_{\mathcal{R},D}$ only costs $|\mathcal{R}| \cdot t$ additions in $\mathbb{K}$.

\begin{algorithm}
\caption{CertificateComp}
\textbf{Input:} \hspace{1cm} \\
order d ∈ \mathbb{Z}^n \\
matrix F ∈ \mathbb{K}[X]^{m \times n} \hspace{2mm} \# \text{such that cdeg(F) < d}, \\
matrix P ∈ \mathbb{K}[X]^{m \times m} \hspace{2mm} \# \text{such that deg(P) ≤ max(d)}.
\textbf{Output:} the coefficient C ∈ \mathbb{K}^{m \times n} of degree 0 of PFX$^{-d}$.
\begin{enumerate}
\item (r$1$, ..., r$m$) ← rdeg(P)
\item C ← 0 ∈ \mathbb{K}^{m \times n}
\item For k from 1 to max(d):
\begin{enumerate}
\item $\mathcal{R} = \{i \in \{1, \ldots, m\} \mid r_i \geq k\}$
\item $\mathcal{D} = \{j \in \{1, \ldots, n\} \mid d_j \geq k\}$
\item $A ∈ \mathbb{K}^{\#\mathcal{R} × \mathcal{D}}$ ← coefficient of degree k of P_{\mathcal{R},s}
\item $B \in \mathbb{K}^{\mathcal{D} × \mathcal{D}}$ ← for all 1 ≤ j ≤ t, $B_{s,j}$ is the coefficient of degree $d_j - k$ of $F_{s,c_j}$
\end{enumerate}
\item C_{\mathcal{R},D} ← C_{\mathcal{R},D} + AB
\item Return C
\end{enumerate}
\end{algorithm}

Consider γ = [rdeg(P)/D] ≥ 1 (indeed, if |rdeg(P)| = 0, then P is constant and C ≡ 0). By Lemma 4.1, at the iteration k we have $|\mathcal{R}| ≤ min(m, yD/k)$ and $t = |\mathcal{D}| ≤ min(n, D/k)$. We separate the cases $n ≤ m$ and $n > m$, and we use the bound $|yD/m| \in O(D/m)$, which comes from our assumptions m ∈ O(D) and γ ∈ O(1).

First, suppose $n ≤ m$. At the iterations $k < [yD/m]$ the matrices A and B both have dimensions at most $m \times m$, hence their product can be computed in O(m$^{e-1}$) operations. These iterations have a total cost of O(m$^{e-1}$([yD/m]) ≤ O(m$^{e-1}$D). At the iterations $k ≥ [yD/m]$, A and B have dimensions at most $(yD/k) \times m$ and $m \times (D/k)$, with $D/k ≤ yD/k ≤ m$; computing their product costs $O(D/k)^{e-1} \leq O(mD^{e-1}k^{1-e})$. Thus, the total cost for these iterations is in

$$O\left(\sum_{k=[yD/m]}^{\max(d)} k^{1-e}\right) \leq O\left(mD^{e-1}([yD/m])^{2-e}\sum_{i=0}^{\infty} \frac{1}{i(2-e)}\right) \leq O(m^{e-1}D).$$

For the first inclusion, we apply Lemma 4.3 with $\mu = [yD/m], \nu = max(d), \omega = 1 − e$. For the second, the sum is finite since $2^{2-e} < 1$. Hence Algorithm 3 costs O(m$^{e-1}$D) in the case $n ≤ m$.

Now, suppose $n > m$. At the iterations $k < [D/n]$, A and B have dimensions at most $m \times m$ and $m \times n$, hence their product can be computed in O(m$^{e-1}$n) operations. The total cost is in O(m$^{e-1}$D) since there are $[D/n] - 1 < D/n$ iterations (with $n ≤ D$ by definition). For the iterations $k ≥ [yD/m]$, we repeat the analysis done above for the same values of $k$: these iterations cost O(m$^{e-1}$D) here as well.

Finally, for the iterations $[D/n] ≤ k < [yD/m]$, A and B have dimensions at most $m \times m$ and $m \times (D/k)$, with $D/k ≤ n$. Thus the product AB can be computed in O(m$^{e-1}$ + m$^{e-1}$D) operations. Summing the term m$^{e-1}$ over these O(D/m) iterations yields the cost O(m$^{e-1}$D). Summing the other term gives the cost O(m$^{e-1}$D log(n/m)) since, by the last claim of Lemma 4.3, we have

$$\sum_{k=[D/n]}^{[yD/m]-1} k^{1-e} \leq 1 + \left|\log_2\left(\frac{|yD/m|}{|D/n|}\right)\right| \leq 1 + \log_2(y/n/m).$$

Adding the costs of the three considered sets of iterations, we obtain the announced cost for Algorithm 3 in the case $n > m$ as well. \hfill \Box
Lemma 4.3. Given integers $0 < \mu < v$ and a real number $\theta \leq 0$,
\[
\sum_{k=\mu}^{v} k^\theta \leq \mu^{\theta+1} \sum_{i=0}^{\ell-1} i^{2(\theta+1)},
\]
holds, where $\ell = \lfloor \log_2(v/\mu) \rfloor + 1$. In particular, $\sum_{k=\mu}^{v} k^{-1} \leq \ell$.

Proof. Note that $\ell$ is chosen such that $2\ell \mu - 1 \geq v$. Then, the upper bound is obtained by splitting the sum as follows:
\[
\sum_{k=\mu}^{v} k^\theta \leq \sum_{i=0}^{\ell-1} \sum_{k=2i\mu}^{2i+1\mu} k^\theta \leq \sum_{i=0}^{\ell-1} \sum_{k=2i\mu}^{2i+1\mu} (2i\mu)^\theta = \sum_{i=0}^{\ell-1} (2i\mu)^{\theta+1},
\]
where the second inequality comes from the fact that $x \mapsto x^\theta$ is decreasing on the positive real numbers.

Finally, we describe minor changes in Algorithm 3 to deal with the case of small average column degree $\text{cd}eg(P) \in O(D)$; precisely, we replace the assumption $|\text{cd}eg(P)| \leq \gamma D$ by $|\text{cd}eg(P)| \leq \gamma D$. Then, instead of the set $\mathbb{R}_k$ used above, we rather define
\[
C_k = \{ j \in \{1, \ldots, m \} \mid \text{cd}eg(P_{x,j}) \geq k \}.
\]
Then we have the following lemma, analogous to Lemma 4.1.

Lemma 4.4. For $k \in \{1, \ldots, m\}$ and $j \notin C_k$, the column $j$ of $P_k$ is zero. In particular, $P_k$ has at most $\#C_k \leq \gamma D / k$ nonzero columns.

Thus, we can modify Algorithm 3 to take into account the column degree of $P$ instead of its row degree. This essentially amounts to redefining the matrices $A$ and $B$ in the loop as follows:

- $A \in \mathbb{R}^{m \times \#C_k}$ is the coefficient of degree $k$ of $P_k C_k$.
- $B \in \mathbb{R}^{\#C_k \times t}$ is such that for all $i \in C_k$ and $1 \leq j \leq t$, $B_{i,j}$ is the coefficient of degree $d_j - k$ of $P_k C_{i,j}$.

These modifications have obviously no impact on the correctness. Furthermore, it is easily verified that the same cost bound holds since we obtain a similar matrix multiplication cost at each iteration.

5 PERSPECTIVES

As noted in the introduction, our certificate is almost optimal since we can verify it at a cost $O(mD + m^{e+1}(m + n))$ while the input size is $mD$. One should notice that the extra term $O(m^{e+1}(m + n))$ corresponds to certifying problems of linear algebra over $\mathbb{R}$, namely the rank and the determinant. These could actually be dealt with in $O(m(m + n))$ operations using interactive certificates built upon the results in [7, 9, 18], thus yielding an optimal certificate. Still, for practical applications, our simpler certification should already be significantly faster than the approximant basis computation, since the constants involved in the cost are small as we have observed in our estimates above. We plan to confirm this for the approximant bases implementations in the LinBox library.

Finally, our verification protocol needs $(m + 2)/\log_2(#S)$ random bits, yielding a probability of failure less than $2^{-\#S}$. The majority of these bits is required by Algorithm 2 when choosing random elements for the vector $u$. As proposed in [19], it may be worthwhile to pick a single random value $\zeta$ and to use $u = [1 \zeta \cdots \zeta^{m-1}]$. In the case where $\max(d) < D/2$, this choice would not affect the probability of failure while decreasing the number of random bits to $3 \log_2(#S)$. In particular, at the price of the same number of bits as we currently use in our algorithm, we could run our verification $(m + 2)/3$ times and decrease the probability of failure to $(D/3)^{2\mu}/3^\mu$.

REFERENCES