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Computing Popov and Hermite Forms of Rectangular Polynomial Matrices

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ABSTRACT

We consider the computation of two normal forms for matrices over the univariate polynomials: the Popov form and the Hermite form. For matrices which are square and nonsingular, deterministic algorithms with satisfactory cost bounds are known. Here, we present deterministic, fast algorithms for rectangular input matrices. The obtained cost bound for the Popov form matches the previous best known randomized algorithm, while the cost bound for the Hermite form improves on the previous best known ones by a factor which is at least the largest dimension of the input matrix.

KEYWORDS

Polynomial matrix; Reduced form; Popov form; Hermite form.

1 INTRODUCTION

In this paper we deal with (univariate) polynomial matrices, i.e. matrices in $\mathbb{K}[x]^{m \times n}$ where \mathbb{K} is a field admitting exact computation, typically a finite field. Given such an input matrix whose row space is the real object of interest, one may ask for a “better” basis for the row space, that is, another matrix which has the same row space but also has additional useful properties. Two important normal forms for such bases are the Popov form [21] and the Hermite form [11], whose definitions are recalled in this paper. The Popov form has rows which have the minimal possible degrees, while the Hermite form is in echelon form. A classical generalisation is the *shifted* Popov form of a matrix [1], where one incorporates degree weights on the columns: with zero shift this is the Popov form, while under some extremal shift this becomes the Hermite form [2]. We are interested in the efficient computation of these forms, which has been studied extensively along with the computation of the related but non-unique reduced forms [6, 13] and weak Popov forms [16].

Hereafter, complexity estimates count basic arithmetic operations in \mathbb{K} on an algebraic RAM, and asymptotic cost bounds omit

factors that are logarithmic in the input parameters, denoted by $O(\cdot)$. We let $2 \leq \omega \leq 3$ be an exponent for matrix multiplication: two matrices in $\mathbb{K}^{m \times m}$ can be multiplied in $O(m^\omega)$ operations. As shown in [5], the multiplication of two polynomials in $\mathbb{K}[x]$ of degree at most d can be done in $O(d)$ operations, and more generally the multiplication of two polynomial matrices in $\mathbb{K}[x]^{m \times m}$ of degree at most d uses $O(m^\omega d)$ operations.

Consider a square, nonsingular $\mathbf{M} \in \mathbb{K}[x]^{m \times m}$ of degree d . For the computation of a reduced form of \mathbf{M} , the complexity $O(m^\omega d)$ was first achieved by a Las Vegas algorithm of Giorgi et al. [7]. All the subsequent work mentioned in the next paragraph achieved the same cost bound, which was taken as a target: up to logarithmic factors, it is the same as the cost for multiplying two matrices with dimensions and degree similar to those of \mathbf{M} .

The approach of [7] was de-randomized by Gupta et al. [9], while Sarkar and Storjohann [23] showed how to compute the Popov form from a reduced form; combining these results gives a deterministic algorithm for the Popov form. Gupta and Storjohann [8, 10] gave a Las Vegas algorithm for the Hermite form; a Las Vegas method for computing the shifted Popov form for any shift was described in [18]. Then, a deterministic Hermite form algorithm was given by Labahn et al. [14], which was one ingredient in a deterministic algorithm due to Neiger and Vu [19] for the arbitrary shift case.

The Popov form algorithms usually exploit the fact that, by definition, this form has degree at most $d = \deg(\mathbf{M})$. While no similarly strong degree bound holds for shifted Popov forms in general (including the Hermite form), these forms still share a remarkable property in the square, nonsingular case: each entry outside the diagonal has degree less than the entry on the diagonal in the same column. These diagonal entries are called *pivots* [13]. Furthermore, their degrees sum to $\deg(\det(\mathbf{M})) \leq md$, so that these forms can be represented with $O(m^2 d)$ field elements, just like \mathbf{M} . This is especially helpful in the design of fast algorithms since this provides ways to control the degrees of the manipulated matrices.

These degree constraints exist but become weaker in the case of *rectangular* shifted Popov forms, say $m \times n$ with $m < n$. Such a normal form does have m columns containing pivots, whose average degree is at most the degree d of the input matrix \mathbf{M} . Yet it also contains $n - m$ columns without pivots, which may all have large degree: up to $\Theta(md)$ in the case of the Hermite form. As a result, a dense representation of the latter form may require $\Omega(m^2(n - m)d)$ field elements, a factor of m larger than for \mathbf{M} . Take for example some $\mathbf{U} \in \mathbb{K}[x]^{m \times m}$ of degree d which is unimodular, meaning that \mathbf{U}^{-1} has entries in $\mathbb{K}[x]$. Then, the Hermite form of $[\mathbf{U} \ \mathbf{I}_m \ \cdots \ \mathbf{I}_m]$ is $[\mathbf{I}_m \ \mathbf{U}^{-1} \ \cdots \ \mathbf{U}^{-1}]$, and the entries of \mathbf{U}^{-1} may have degree in $\Omega(md)$. However, the Popov form, having minimal degree, has size in $O(mnd)$, just like \mathbf{M} . Thus, unlike in the nonsingular case, one

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would set different target costs for the computation of Popov and Hermite forms, such as $O(m^{\omega-1}nd)$ for the former and $O(m^\omega nd)$ for the latter (note that the exponent affects the small dimension).

For a rectangular matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$, Mulders and Storjohann [16] gave an iterative Popov form algorithm which costs $O(rmnd^2)$, where r is the rank of \mathbf{M} . Beckermann et al. [3] obtain the shifted Popov form for any shift by computing a basis of the left kernel of $[\mathbf{M}^\top \ \mathbf{I}_n]^\top$. This approach also produces a matrix which transforms \mathbf{M} into its normal form and whose degree can be in $\Omega(md)$: efficient algorithms usually avoid computing this transformation. To find the sought kernel basis, the fastest known method is to compute a shifted Popov approximant basis of the $(m+n) \times n$ matrix above, at an order which depends on the shift. [3] relies on a fraction-free algorithm for the latter computation, and hence lends itself well to cases where \mathbb{K} is not finite. In our context, following this approach with the fastest known approximant basis algorithm [12] yields the cost bounds $O((m+n)^{\omega-1}nmd)$ for the Popov form and $O((m+n)^{\omega-1}n^2md)$ for the Hermite form. For the latter this is the fastest existing algorithm, to the best of our knowledge.

For \mathbf{M} with full rank and $m \leq n$, Sarkar [22] showed a Las Vegas algorithm for the Popov form achieving the cost $O(m^{\omega-1}nd)$. This uses random column operations to compress \mathbf{M} into an $m \times m$ matrix, which is then transformed into a reduced form. Applying the same transformation on \mathbf{M} yields a reduced form of \mathbf{M} with high probability, and from there the Popov form can be obtained. Lowering this cost further seems difficult, as indicated in the square case by the reduction from polynomial matrix multiplication to Popov form computation described in [23, Thm. 22].

For a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ which is rank-deficient or has $m > n$, the computation of a basis of the row space of \mathbf{M} was handled by Zhou and Labahn [29] with cost $O(m^{\omega-1}(m+n)d)$. Their algorithm is deterministic, and the output basis $\mathbf{B} \in \mathbb{K}[x]^{r \times n}$ has degree at most d . This may be used as a preliminary step: the normal form of \mathbf{M} is also that of \mathbf{B} , and the latter has full rank with $r \leq n$.

We stress that, from a rectangular matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$, it seems difficult in general to predict which columns of its shifted Popov form will be pivot-free. For this reason, there seems to be no obvious deterministic reduction from the rectangular case to the square case, even when n is only slightly larger than m . Sarkar's algorithm is a Las Vegas reduction, *compressing* the matrix to a nonsingular $m \times m$ matrix; another Las Vegas reduction consists in *completing* the matrix to a nonsingular $n \times n$ matrix (see Section 3).

In the nonsingular case, exploiting information on the pivots has led to algorithmic improvements for normal form algorithms [10, 12, 14, 23]. Following this, we put our effort into two computational tasks: finding the location of the pivots in the normal form (the *pivot support*), and using this knowledge to compute this form.

Our first contribution is to show how to efficiently find the pivot support of \mathbf{M} . For this we resort to the so-called saturation of \mathbf{M} computed in a form which reveals the pivot support (Section 4.1), making use of an idea from [28]. While this is only efficient for $n \in O(m)$, using this method repeatedly on well-chosen submatrices of \mathbf{M} with about $2m$ columns allows us to find the pivot support using $O(m^{\omega-1}nd)$ operations for any dimensions $m \leq n$ (Section 4.2).

In our second main contribution, we consider the shifted Popov form of \mathbf{M} , for any shift. We show that once its pivot support is

known, then this form can be computed efficiently (Section 6 and Proposition 6.1). In particular, combining both contributions yields a fast and deterministic Popov form algorithm.

THEOREM 1.1. *For a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ of degree at most d and with $m \leq n$, there is a deterministic algorithm which computes the Popov form of \mathbf{M} using $O(m^{\omega-1}nd)$ operations in \mathbb{K} .*

The second contribution may of course be useful in situations where the pivot support is known for some reason. Yet, there are even general cases where it can be computed efficiently, namely when the shift has very unbalanced entries. This is typically the case of the Hermite form, for which the pivot support coincides with the column rank profile of \mathbf{M} . The latter can be efficiently obtained via an algorithm due to Zhou [26, Sec. 11], based on the kernel basis algorithm from [30]. This leads us to the next result.

THEOREM 1.2. *Let $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ with full rank and $m < n$. There is a deterministic algorithm which computes the Hermite form of \mathbf{M} using $O(m^{\omega-1}n\delta)$ operations in \mathbb{K} , where δ is the minimum of the sum of column degrees of \mathbf{M} and of the sum of row degrees of \mathbf{M} .*

Using this quantity δ (see Eq. (6) for a more precise definition), the mentioned cost for the kernel basis approach of [3] becomes $O((m+n)^{\omega-1}n^2\delta)$. Thus, when $n \in O(m)$ the cost in the above theorem already gains a factor n compared to this approach; when n is large compared to m , this factor becomes $n(\frac{n}{m})^{\omega-1}$.

2 PRELIMINARIES

2.1 Basic notation

If \mathbf{M} is an $m \times n$ matrix and $1 \leq j \leq n$, we denote by $\mathbf{M}_{*,j}$ the j th column of \mathbf{M} . If $J \subseteq \{1, \dots, n\}$ is a set of column indices, $\mathbf{M}_{*,J}$ is the submatrix of \mathbf{M} formed by the columns at the indices in J . We use analogous row-wise notation. Similarly, for a tuple $\mathbf{t} \in \mathbb{Z}^n$, then \mathbf{t}_j is the subtuple of \mathbf{t} formed by the entries at the indices in J .

When adding a constant to an integer tuple, for example $\mathbf{t} + 1$ for some $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{Z}^m$, we really mean $(t_1 + 1, \dots, t_m + 1)$; when comparing a tuple to a constant, for example $\mathbf{t} \leq 1$, we mean $\max(\mathbf{t}) \leq 1$. Two tuples of the same length will always be compared entrywise: $\mathbf{s} \leq \mathbf{t}$ stands for $s_i \leq t_i$ for all i . We use the notation $\text{amp}(\mathbf{t}) = \max(\mathbf{t}) - \min(\mathbf{t})$, and $|\mathbf{t}| = t_1 + \dots + t_m$ (note that the latter will mostly be used when \mathbf{t} has nonnegative entries).

For a given nonnegative integer tuple $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{Z}_{\geq 0}^m$, we denote by $\mathbf{x}^{\mathbf{t}}$ the diagonal matrix with entries x^{t_1}, \dots, x^{t_m} .

2.2 Row spaces, kernels, and approximants

For a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$, its *row space* is the $\mathbb{K}[x]$ -module generated by its rows, that is, $\{\lambda \mathbf{M}, \lambda \in \mathbb{K}[x]^{1 \times m}\}$. Then, a matrix $\mathbf{B} \in \mathbb{K}[x]^{r \times n}$ is a *row basis* of \mathbf{M} if its rows form a basis of the row space of \mathbf{M} , in which case r is the rank of \mathbf{M} .

The *left kernel* of \mathbf{M} is the $\mathbb{K}[x]$ -module $\{\mathbf{p} \in \mathbb{K}[x]^{1 \times m} \mid \mathbf{p}\mathbf{M} = \mathbf{0}\}$. A matrix $\mathbf{K} \in \mathbb{K}[x]^{k \times m}$ is a *left kernel basis* of \mathbf{M} if its rows form a basis of this kernel, in which case $k = m - r$. Similarly, a *right kernel basis* of \mathbf{M} is a matrix $\mathbf{K} \in \mathbb{K}[x]^{n \times (n-r)}$ whose *columns* form a basis of the right kernel of \mathbf{M} .

Given $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}_{>0}^n$, the set of *approximants* for \mathbf{M} at order \mathbf{d} is the $\mathbb{K}[x]$ -module of rank m defined as

$$\mathcal{A}_{\mathbf{d}}(\mathbf{M}) = \{\mathbf{p} \in \mathbb{K}[x]^{1 \times m} \mid \mathbf{p}\mathbf{M} = \mathbf{0} \bmod \mathbf{x}^{\mathbf{d}}\}.$$

The identity $\mathbf{pM} = \mathbf{0} \bmod x^d$ means that the j th entry of the vector $\mathbf{pM} \in \mathbb{K}[x]^{1 \times n}$ is divisible by x^d , for all j .

Two $m \times n$ matrices $\mathbf{M}_1, \mathbf{M}_2$ have the same row space if and only if they are *unimodularly equivalent*, that is, there is a unimodular matrix $\mathbf{U} \in \mathbb{K}[x]^{m \times m}$ such that $\mathbf{UM}_1 = \mathbf{M}_2$. For $\mathbf{M}_3 \in \mathbb{K}[x]^{r \times n}$ with $r \leq m$, \mathbf{M}_1 and \mathbf{M}_3 have the same row space exactly when \mathbf{M}_3 padded with $m - r$ zero rows is unimodularly equivalent to \mathbf{M}_1 .

2.3 Row degrees and reduced forms

For a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$, we denote by $\text{rdeg}(\mathbf{M})$ the tuple of the degrees of its rows, that is, $(\text{deg}(\mathbf{M}_{1,*}), \dots, \text{deg}(\mathbf{M}_{m,*}))$.

If \mathbf{M} has no zero row, the (*row-wise*) *leading matrix* of \mathbf{M} , denoted by $\text{lm}(\mathbf{M})$, is the matrix in $\mathbb{K}^{m \times n}$ whose entry i, j is equal to the coefficient of degree $\text{deg}(\mathbf{M}_{i,*})$ of the entry i, j of \mathbf{M} .

For a matrix $\mathbf{R} \in \mathbb{K}[x]^{m \times n}$ with no zero row and $m \leq n$, we say that \mathbf{R} is (*row*) *reduced* if $\text{lm}(\mathbf{R})$ has full rank. Thus, here a reduced matrix must have full rank (and no zero row), as in [6]. For more details about reduced matrices, we refer the reader to [3, 6, 13, 25]. In particular, we have the following characterizing properties:

- *Predictable degree property* [6] [13, Thm. 6.3-13]: we have

$$\text{deg}(\lambda \mathbf{R}) = \max\{\text{deg}(\lambda_i) + \text{rdeg}(\mathbf{R}_{i,*}), 1 \leq i \leq m\}$$

for any vector $\lambda = [\lambda_i]_i \in \mathbb{K}[x]^{1 \times m}$.

- *Minimality of the sum of row degrees* [6]: for any nonsingular matrix $\mathbf{U} \in \mathbb{K}[x]^{m \times m}$, we have $|\text{rdeg}(\mathbf{UR})| \geq |\text{rdeg}(\mathbf{R})|$.
- *Minimality of the tuple of row degrees* [26, Sec. 2.7]: for any nonsingular matrix $\mathbf{U} \in \mathbb{K}[x]^{m \times m}$, we have $s \leq t$ where the tuples s and t are the row degrees of \mathbf{R} and of \mathbf{UR} sorted in nondecreasing order, respectively.

From the last item, it follows that two unimodularly equivalent reduced matrices have the same row degree up to permutation.

For a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$, we call *reduced form* of \mathbf{M} any reduced matrix $\mathbf{R} \in \mathbb{K}[x]^{r \times n}$ which is a row basis of \mathbf{M} . The third item above shows that $\text{deg}(\mathbf{R}) \leq \text{deg}(\mathbf{M})$.

2.4 Pivots and Popov forms

For a nonzero vector $\mathbf{p} = [p_j]_j \in \mathbb{K}[x]^{1 \times m}$, the *pivot index* of \mathbf{p} is the largest index j such that $\text{deg}(p_j) = \text{deg}(\mathbf{p})$ [13, Sec. 6.7.2]. In this case we call p_j the *pivot entry* of \mathbf{p} . For the zero vector, we define its degree to be $-\infty$ and its pivot index to be 0. Further, the *pivot index* of a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ is the tuple $(j_1, \dots, j_m) \in \mathbb{Z}_{\geq 0}^m$ such that j_i is the pivot index of $\mathbf{M}_{i,*}$. Note that we will only use the word “pivot” in this row-wise sense.

A matrix $\mathbf{P} \in \mathbb{K}[x]^{m \times n}$ is in *weak Popov form* if it has no zero row and the entries of the pivot index of \mathbf{P} are all distinct [16]; a weak Popov form is further called *ordered* if its pivot index is (strictly) increasing order. A weak Popov matrix is also reduced.

The (ordered) weak Popov form is not canonical: a given row space may have many (ordered) weak Popov forms. The Popov form adds a normalization property, yielding a canonical form; we use the definition from [2, Def. 3.3]:

A matrix $\mathbf{P} \in \mathbb{K}[x]^{m \times n}$ is in *Popov form* if it is in ordered weak Popov form, the corresponding pivot entries are monic, and in each column of \mathbf{P} which contains a (row-wise) pivot the other entries have degree less than this pivot entry.

For a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ of rank r , there exists a unique $\mathbf{P} \in \mathbb{K}[x]^{r \times n}$ which is in Popov form and has the same row space as \mathbf{M} [3, Thm. 2.7]. We call \mathbf{P} the *Popov form* of \mathbf{M} . For a more detailed treatment of Popov forms, see [2, 3, 13].

For example, consider the unimodularly equivalent matrices

$$\begin{bmatrix} x^2 & x+1 & 2 \\ 2x+2 & 2x & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x^2 - x - 1 & 1 & 1 \\ x+1 & x & 1 \end{bmatrix},$$

defined over $\mathbb{F}_7[x]$; the first one is in weak Popov form and the second one is its Popov form. Note that any deterministic rule for ordering the rows would lead to a canonical form; we use that of [2, 3], while that of [13, 16] sorts the rows by degrees and would consider the second matrix not to be normalized.

Going back to the general case, we denote by $\pi(\mathbf{M}) \in \mathbb{Z}_{>0}^r$ the pivot index of the Popov form of \mathbf{M} , called the *pivot support* of \mathbf{M} . In most cases, $\pi(\mathbf{M})$ differs from the pivot index of \mathbf{M} . We have the following important properties:

- The pivot index of \mathbf{M} is equal to the pivot support $\pi(\mathbf{M})$ and only if \mathbf{M} is in ordered weak Popov form.
- For any $\lambda \in \mathbb{K}[x]^{1 \times m}$ such that $\lambda \mathbf{M} \neq \mathbf{0}$, the pivot index of $\lambda \mathbf{M}$ appears in the pivot support $\pi(\mathbf{M})$; in particular each nonzero entry of the pivot index of \mathbf{M} is in $\pi(\mathbf{M})$.

For the first item, we refer to [3, Sec. 2] (in this reference, the set formed by the entries of the pivot support is called “pivot set” and ordered weak Popov forms are called quasi-Popov forms). The second item is a simple extension of the predictable degree property (see for example [17, Lem. 1.17] for a proof).

2.5 Computational tools

We will rely on the following result from [30, Cor. 4.6 and Thm. 3.4] about the computation of kernel bases in reduced form. Note that a matrix is *column reduced* if its transpose is reduced.

THEOREM 2.1 ([30]). *There is an algorithm MINIMALKERNELBASIS which, given a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ with $m \leq n$, returns a right kernel basis $\mathbf{K} \in \mathbb{K}[x]^{m \times (n-r)}$ of \mathbf{M} in column reduced form using*

$$\tilde{O}(n^\omega [m \text{ deg}(\mathbf{M})/n]) \subseteq \tilde{O}(n^\omega \text{deg}(\mathbf{M}))$$

operations in \mathbb{K} . Furthermore, $|\text{cdeg}(\mathbf{K})| \leq r \text{deg}(\mathbf{M})$.

For the computation of normal forms of square, nonsingular matrices, we use the next result (*s*-Popov forms will be introduced in Section 5; Popov forms as above correspond to $s = 0$).

THEOREM 2.2 ([19]). *There is an algorithm NONSINGULARPOPOV which, given a nonsingular matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times m}$ and a shift $s \in \mathbb{Z}^m$, returns the *s*-Popov form of \mathbf{M} using*

$$\tilde{O}(m^\omega \lceil |\text{rdeg}(\mathbf{M})|/m \rceil) \subseteq \tilde{O}(m^\omega \text{deg}(\mathbf{M}))$$

operations in \mathbb{K} .

This is [19, Thm. 1.3] with a minor modification: we have replaced the so-called generic determinant bound by a larger quantity (the sum of row degrees), since this is sufficient for our needs here.

3 POPOV FORM VIA COMPLETION INTO A SQUARE AND NONSINGULAR MATRIX

We now present a new Las Vegas algorithm for computing the (non-shifted) Popov form \mathbf{P} of a rectangular matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ with

full rank and $m < n$, relying on algorithms for the case of square, nonsingular matrices. In the case $n \in O(m)$, this results in a cost bounded by $O(m^\omega \deg(\mathbf{M}))$, which has already been obtained by the Las Vegas algorithm of Sarkar [22]; however, the advantage of our approach is that it becomes asymptotically faster if the *average* row degree of \mathbf{M} is significantly smaller than $\deg(\mathbf{M})$.

The idea is to find a matrix $\mathbf{C} \in \mathbb{K}[x]^{(n-m) \times n}$ such that the Popov form of $[\mathbf{M}^\top \ \mathbf{C}^\top]^\top$ contains \mathbf{P} as an identifiable subset of its rows. We will show that if \mathbf{C} is drawn randomly of sufficiently high degree, then this is true with high probability.

Definition 3.1. Let $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ have full rank with $m < n$ and let $\mathbf{P} \in \mathbb{K}[x]^{m \times n}$ be the Popov form of \mathbf{M} . A *completion* of \mathbf{M} is any matrix $\mathbf{C} \in \mathbb{K}[x]^{(n-m) \times n}$ such that:

$$\min(\text{rdeg}(\mathbf{C})) > \deg(\mathbf{P}) \text{ and } \begin{bmatrix} \mathbf{P} \\ \mathbf{C} \end{bmatrix} \text{ is row reduced.}$$

The next lemma shows that: 1) if \mathbf{C} is a completion, then \mathbf{P} will appear as a submatrix of the Popov form of $[\mathbf{M}^\top \ \mathbf{C}^\top]^\top$; and 2) we can easily check from that Popov form whether \mathbf{C} is a completion or not. The latter is essential for a Las Vegas algorithm.

Lemma 3.2. Let $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ have full rank with $m < n$ with Popov form \mathbf{P} , and let $\mathbf{C} \in \mathbb{K}[x]^{(n-m) \times n}$ be such that $[\mathbf{M}^\top \ \mathbf{C}^\top]^\top$ has full rank and $\min(\text{rdeg}(\mathbf{C})) > \deg(\mathbf{P})$. Then, \mathbf{C} is a completion of \mathbf{M} if and only if $\text{rdeg}(\hat{\mathbf{P}})$ contains a permutation of $\text{rdeg}(\mathbf{C})$, where $\hat{\mathbf{P}}$ is the Popov form of $[\mathbf{M}^\top \ \mathbf{C}^\top]^\top$. In this case, \mathbf{P} is the submatrix of $\hat{\mathbf{P}}$ formed by its rows of degree less than $\min(\text{rdeg}(\mathbf{C}))$.

PROOF. First, we assume that \mathbf{C} is a completion of \mathbf{M} . Then $[\mathbf{P}^\top \ \mathbf{C}^\top]^\top$ is reduced, and therefore it has the same row degree as its Popov form $\hat{\mathbf{P}}$ up to permutation. Hence, in particular, $\text{rdeg}(\hat{\mathbf{P}})$ contains a permutation of $\text{rdeg}(\mathbf{C})$.

Now, we assume that $\text{rdeg}(\hat{\mathbf{P}})$ contains a permutation of $\text{rdeg}(\mathbf{C})$ and our goal is to show that $[\mathbf{P}^\top \ \mathbf{C}^\top]^\top$ is reduced and $\hat{\mathbf{P}}$ contains \mathbf{P} as a submatrix. Let $\hat{\mathbf{P}}_1$ be the submatrix of $\hat{\mathbf{P}}$ of its rows of degree less than $\min(\text{rdeg}(\mathbf{C}))$; and $\hat{\mathbf{P}}_2$ be the submatrix of the remaining rows. By assumption, $\hat{\mathbf{P}}_2$ has at least $n - m$ rows and $\hat{\mathbf{P}}_1$ has at most m rows. Since $\hat{\mathbf{P}}$ is also the Popov form of $[\mathbf{P}^\top \ \mathbf{C}^\top]^\top$, there is a unimodular transformation

$$\begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{P}}_1 \\ \hat{\mathbf{P}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{P} \\ \mathbf{C} \end{bmatrix}. \quad (1)$$

By the predictable degree property we obtain $\mathbf{U}_{12} = \mathbf{0}$; thus, since \mathbf{P} has full rank m , then $\hat{\mathbf{P}}_1$ has exactly m rows, and \mathbf{U}_{11} is unimodular. Therefore $\hat{\mathbf{P}}_1 = \mathbf{P}$ since both matrices are in Popov form. As a result, $\text{rdeg}(\hat{\mathbf{P}})$ is a permutation of $(\text{rdeg}(\mathbf{P}), \text{rdeg}(\mathbf{C}))$. \square

Lemma 3.3. Let $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ have full rank with $m < n$. Let $S \subseteq \mathbb{K}$ be finite of cardinality q and let $\mathbf{L} \in \mathbb{K}^{(n-m) \times n}$ with entries chosen independently and uniformly at random from S . Then $x^{\deg(\mathbf{M})+1}\mathbf{L}$ is a completion of \mathbf{M} with probability at least $\prod_{i=1}^{n-m} (1 - q^{-i})$ if \mathbb{K} is finite and $S = \mathbb{K}$, and at least $1 - \frac{n-m}{q}$ otherwise.

PROOF. Let $d = \deg(\mathbf{M})$. We first note that for $x^{d+1}\mathbf{L}$ to be a completion of \mathbf{M} , it is enough that the matrix

$$\text{lm} \left(\begin{bmatrix} \mathbf{P} \\ \mathbf{C} \end{bmatrix} \right) = \begin{bmatrix} \text{lm}(\mathbf{P}) \\ \text{lm}(\mathbf{C}) \end{bmatrix} = \begin{bmatrix} \text{lm}(\mathbf{P}) \\ \mathbf{L} \end{bmatrix} \in \mathbb{K}^{n \times n}$$

be invertible. Indeed, this implies first that $[\mathbf{P}^\top \ \mathbf{C}^\top]^\top$ is reduced; and second, that \mathbf{C} has no zero row, hence $\text{rdeg}(\mathbf{C}) = (d+1, \dots, d+1)$ and $\min(\text{rdeg}(\mathbf{C})) = d+1 > \deg(\mathbf{M}) \geq \deg(\mathbf{P})$.

In the case of a finite field \mathbb{K} with q elements, the probability that the above matrix is invertible is $\prod_{i=1}^{n-m} (1 - q^{-i})$. If \mathbb{K} is infinite or of cardinality $\geq q$, the Schwartz-Zippel lemma implies that the probability that the above matrix is singular is at most $(n-m)/q$. \square

Thus, if \mathbb{K} is infinite, it is sufficient to take S of cardinality at least $2(n-m)$ to ensure that $x^{d+1}\mathbf{L}$ is a completion with probability at least $1/2$. On the other hand, if \mathbb{K} is finite of cardinality q , we have the following bounds on the probability:

$$\prod_{i=1}^{n-m} (1 - q^{-i}) > \begin{cases} 0.28 & \text{if } q = 2, \\ 0.55 & \text{if } q = 3, \\ 0.75 & \text{if } q > 5. \end{cases}$$

In Algorithm 1, we first test the nonsingularity of $\mathbf{N} = [\mathbf{M}^\top \ \mathbf{C}^\top]^\top$ before computing $\hat{\mathbf{P}}$, since the fastest known Popov form algorithms in the square case do not support singular matrices. Over a field with at least $2n \deg(\mathbf{N}) + 1$ elements, a simple Monte Carlo test for this is to evaluate the polynomial matrix at a random $\alpha \in \mathbb{K}$ and testing the resulting scalar matrix for nonsingularity; this falsely reports singularity only if $\det(\mathbf{N})$ is divisible by $(x - \alpha)$. Alternatively, a deterministic check is as follows. First, apply the partial linearization of [9, Sec. 6], yielding a matrix $\bar{\mathbf{N}} \in \mathbb{K}[x]^{n \times \bar{n}}$ such that $\bar{\mathbf{N}}$ is nonsingular if and only if \mathbf{N} is nonsingular; $\bar{n} \in O(n)$; and $\deg(\bar{\mathbf{N}}) \leq \lceil \text{rdeg}(\mathbf{N})/n \rceil$. This does not involve arithmetic operations. Since $\bar{\mathbf{N}}$ is nonsingular if and only if its kernel is trivial, it then remains to compute a kernel basis via the algorithm in [27], using $O(n^\omega \deg(\bar{\mathbf{N}})) \subseteq O(n^\omega \lceil \text{rdeg}(\mathbf{N})/n \rceil)$ operations in \mathbb{K} . Instead of considering the kernel, one could also test the nonsingularity of $\bar{\mathbf{N}}$ using algorithms from [9], as explained in [22, p. 24].

Algorithm 1: RANDOMCOMPLETIONPOPOV

Input: matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ with full rank and $m < n$; subset $S \subseteq \mathbb{K}$ of cardinality q .

Output: the Popov form of \mathbf{M} , or *failure*.

1. $\mathbf{L} \leftarrow$ matrix in $\mathbb{K}^{(n-m) \times n}$ with entries chosen uniformly and independently at random from S .
2. $\mathbf{C} \leftarrow x^{\deg(\mathbf{M})+1}\mathbf{L}$
3. *If* $[\mathbf{M}^\top \ \mathbf{C}^\top]^\top$ *is singular then return failure*
4. $\hat{\mathbf{P}} \leftarrow \text{NONSINGULARPOPOV}([\mathbf{M}^\top \ \mathbf{C}^\top]^\top)$
5. *If* $\text{rdeg}(\hat{\mathbf{P}})$ *does not contain a permutation of* $\text{rdeg}(\mathbf{C})$ *then return failure*
6. *Return* the submatrix of $\hat{\mathbf{P}}$ *formed by its rows of degree less than* $\min(\text{rdeg}(\mathbf{C}))$

PROPOSITION 3.4. Algorithm 1 is correct and the probability that a failure is reported at Step 3 or Step 5 is as indicated in Lemma 3.3. *If* NONSINGULARPOPOV *is the algorithm of [19], Algorithm 1 uses*

$$\tilde{O} \left(n^\omega \left\lceil \frac{|\text{rdeg}(\mathbf{M})| + (n-m) \deg(\mathbf{M})}{n} \right\rceil \right) \subseteq \tilde{O}(n^\omega \deg(\mathbf{M}))$$

operations in \mathbb{K} .

Indeed, from Theorem 2.2, Step 4 uses $O(n^\omega \lceil \Delta/n \rceil)$ operations where $\Delta = |\text{rdeg}([\mathbf{M}^\top \ \mathbf{C}^\top]^\top)| = |\text{rdeg}(\mathbf{M})| + (n-m)(\text{deg}(\mathbf{M}) + 1)$.

While other Popov form algorithms could be used, that of [19] allows us to take into account the average row degree of \mathbf{M} . Indeed, if $|\text{rdeg}(\mathbf{M})| \ll m \text{deg}(\mathbf{M})$ and $n - m \ll n$, the cost bound above is asymptotically better than $O(n^\omega \text{deg}(\mathbf{M}))$.

Remark 1: As we mentioned in Section 2.4, the pivot index of \mathbf{M} is a subset of $\pi(\mathbf{M})$. Therefore, one can let \mathbf{L} be zero at all columns where \mathbf{M} has a pivot, or indices one otherwise knows appear in $\pi(\mathbf{M})$. If \mathbf{M} has uneven degrees (e.g. it has the form $\hat{\mathbf{M}}\mathbf{x}^s$ for some shift s , see Section 5.1), then this can be particularly worthwhile. In the case where for some reason we know $\pi(\mathbf{M})$, then \mathbf{L} can simply be taken such that $\mathbf{L}_{*, \{1, \dots, n\} \setminus \pi(\mathbf{M})}$ is the identity matrix. In that case, Algorithm 1 becomes deterministic.

4 COMPUTING THE PIVOT SUPPORT

We now consider a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ with $m < n$, possibly rank-deficient, and we focus on the computation of its pivot support $\pi(\mathbf{M})$. In Section 4.1, we give a deterministic algorithm which is efficient when $n \in O(m)$. In Section 4.2 we explain how this can be used iteratively to efficiently find the pivot support when $m \ll n$.

4.1 Deterministic pivot support computation via column basis factorization

Our approach stems from the fact (see Lemma 4.2) that $\pi(\mathbf{M})$ is also the pivot support of any basis of the *saturation* of the row space of \mathbf{M} [4, Sec. II.§2.4], defined as

$$\{\lambda \mathbf{M}, \lambda \in \mathbb{K}(x)^{1 \times m}\} \cap \mathbb{K}[x]^{1 \times m}.$$

This notion of saturation was already used in [28] in order to compute column bases of \mathbf{M} by relying on the following factorization:

LEMMA 4.1 ([28, SEC. 3]). *Let $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ have rank $r \in \mathbb{Z}_{>0}$, let $\mathbf{K} \in \mathbb{K}[x]^{n \times (n-r)}$ be a right kernel basis of \mathbf{M} , and let $\mathbf{S} \in \mathbb{K}[x]^{r \times n}$ be a left kernel basis of \mathbf{K} . Then, we have $\mathbf{M} = \mathbf{C}\mathbf{S}$ for some column basis $\mathbf{C} \in \mathbb{K}[x]^{m \times r}$ of \mathbf{M} .*

One can easily verify that the left kernel of \mathbf{K} is precisely the saturation of \mathbf{M} , and therefore the matrix \mathbf{S} is a (row) basis of this saturation. Here, we are particularly interested in the following consequence of this result:

LEMMA 4.2. *The matrices \mathbf{M} and \mathbf{S} in Lemma 4.1 have the same pivot support, that is, $\pi(\mathbf{M}) = \pi(\mathbf{S})$.*

PROOF. Since $\mathbf{M} = \mathbf{C}\mathbf{S}$, the row space of \mathbf{M} is contained in that of \mathbf{S} . Hence, by the properties at the end of Section 2.4, $\pi(\mathbf{M}) \subseteq \pi(\mathbf{S})$ as sets. But since \mathbf{M} and \mathbf{S} both have rank r , both pivot supports have exactly r different elements, and must be equal. \square

We will read off $\pi(\mathbf{S})$ from \mathbf{S} by ensuring that this matrix is in ordered weak Popov form. First, we obtain a column reduced right kernel basis \mathbf{K} of \mathbf{M} using `MINIMALKERNELBASIS` (see Theorem 2.1). However, the degree profile of \mathbf{K} prevents us from using the same algorithm to compute a left kernel basis \mathbf{S} efficiently, since the average row degree of \mathbf{K} could be as large as $r \text{deg}(\mathbf{M})$. To circumvent this issue, we combine the observations that $\text{deg}(\mathbf{S})$ is bounded and that \mathbf{K} has small average *column* degree to conclude that \mathbf{S} can be efficiently obtained via an approximant basis (see Section 2).

LEMMA 4.3. *Let $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ have rank $r \in \mathbb{Z}_{>0}$ and let $\mathbf{K} \in \mathbb{K}[x]^{n \times (n-r)}$ be a right kernel basis of \mathbf{M} . Then, any left kernel basis of \mathbf{K} which is in reduced form must have degree at most $d = \text{deg}(\mathbf{M})$. As a consequence, if $\hat{\mathbf{P}} \in \mathbb{K}[x]^{n \times n}$ is a reduced basis of $\mathcal{A}_d(\mathbf{K})$, where $d = \text{cdeg}(\mathbf{K}) + d + 1 \in \mathbb{Z}^{n-r}$, then the submatrix \mathbf{P} of $\hat{\mathbf{P}}$ formed by its rows of degree at most d is a reduced left kernel basis of \mathbf{K} .*

PROOF. Let $\mathbf{S} \in \mathbb{K}[x]^{r \times n}$ be a left kernel basis of \mathbf{K} in reduced form. By Lemma 4.1, $\mathbf{M} = \mathbf{C}\mathbf{S}$ for some matrix $\mathbf{C} \in \mathbb{K}[x]^{m \times r}$. Then, the predictable degree property implies that $\text{deg}(\mathbf{S}) \leq \text{deg}(\mathbf{C}\mathbf{S}) = d$.

For the second claim (which is a particular case of [28, Lem. 4.2]), note that \mathbf{P} is reduced as a subset of the rows of a reduced matrix. Besides, $\text{cdeg}(\mathbf{P}\mathbf{K}) < d$ by construction, hence $\mathbf{P}\mathbf{K} = \mathbf{0} \bmod \mathbf{x}^d$ implies $\mathbf{P}\mathbf{K} = \mathbf{0}$. It remains to show that \mathbf{P} generates the left kernel of \mathbf{K} . Indeed, there exists a basis of this kernel which has degree at most d , and on the other hand any vector of degree at most d in this kernel is in particular in $\mathcal{A}_d(\mathbf{K})$ and therefore is a combination of the rows of $\hat{\mathbf{P}}$; using the predictable degree property, we obtain that this combination only involves rows from the submatrix \mathbf{P} . \square

If we compute $\hat{\mathbf{P}}$ in ordered weak Popov form, then the submatrix \mathbf{P} is in ordered weak Popov form as well, and therefore $\pi(\mathbf{M})$ can be directly read off from it. The computation of an approximant basis in ordered weak Popov form can be done via the algorithm of [12], which returns one in Popov form.

Algorithm 2: `PIVOTSUPPORTVIAFACTOR`

Input: matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ with $m \leq n$.

Output: the pivot support $\pi(\mathbf{M})$ of \mathbf{M} .

1. If $\mathbf{M} = \mathbf{0}$ then return the empty tuple $() \in \mathbb{Z}_{>0}^0$
2. $\mathbf{K} \in \mathbb{K}[x]^{n \times (n-r)} \leftarrow \text{MINIMALKERNELBASIS}(\mathbf{M})$
3. $\hat{\mathbf{P}} \in \mathbb{K}[x]^{n \times n} \leftarrow$ ordered weak Popov basis of $\mathcal{A}_d(\mathbf{K})$, with $d = \text{cdeg}(\mathbf{K}) + (\text{deg}(\mathbf{M}) + 1) \in \mathbb{Z}^{n-r}$
4. $\mathbf{S} \in \mathbb{K}[x]^{r \times n} \leftarrow$ the rows of $\hat{\mathbf{P}}$ of degree at most $\text{deg}(\mathbf{M})$
5. Return the pivot index of \mathbf{S}

PROPOSITION 4.4. *Algorithm 2 is correct and uses $O(n^\omega \text{deg}(\mathbf{M}))$ operations in \mathbb{K} .*

PROOF. Note that we compute the rank of \mathbf{M} as r by the indirect assignment at Step 2. Besides, \mathbf{S} is in ordered weak Popov form since it is a submatrix formed by rows of $\hat{\mathbf{P}}$ itself in ordered weak Popov form. This implies that Step 5 indeed returns the pivot support of \mathbf{S} . Then, the correctness directly follows from Lemmas 4.2 and 4.3.

By Theorem 2.1, Step 2 costs $O(n^\omega d)$, where $d = \text{deg}(\mathbf{M})$, and $|\text{cdeg}(\mathbf{K})| \leq rd$. Thus, the sum of the approximation order defined at Step 3 is $|d| = |\text{cdeg}(\mathbf{K})| + (n-r)(d+1) < n(d+1)$. Then, this step uses $O(n^{\omega-1}|d|) \subseteq O(n^\omega d)$ operations [12, Thm. 1.4]. \square

Note that in this algorithm we do not require that \mathbf{M} has full rank. The only reason why we assume $m \leq n$ is because the cost bound for the computation of a kernel basis at Step 2 is not clear to us in the case $m > n$ (the same assumption is made in [30]).

Here, it seems more difficult to take average degrees into account than in Algorithm 1. While the average degree of the m columns of \mathbf{M} with largest degree could be taken into account by the kernel basis algorithm of [30], it seems that the computation of \mathbf{S} via an approximant basis remains in $O(n^\omega d)$ nevertheless.

4.2 The case of wide matrices

In this section we will deal with pivots of submatrices $\mathbf{M}_{*,J}$, where $J = \{j_1 < \dots < j_k\} \subseteq \{1, \dots, n\}$. To use column indices of $\mathbf{M}_{*,J}$ in \mathbf{M} , we introduce for any such J the operator $\phi_J : \{1, \dots, k\} \rightarrow \{1, \dots, n\}$ satisfying $\phi_J(i) = j_i$. We abuse notation by applying ϕ_J element-wise to tuples, such as in $\phi_J(\boldsymbol{\pi}(\mathbf{M}_{*,J}))$.

The following simple lemma is the crux of the algorithm:

LEMMA 4.5. *Let $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$, and consider any set of indices $J \subseteq \{1, \dots, n\}$. Then $(\boldsymbol{\pi}(\mathbf{M}) \cap J) \subseteq \phi_J(\boldsymbol{\pi}(\mathbf{M}_{*,J}))$ with equality whenever $\boldsymbol{\pi}(\mathbf{M}) \subseteq J$.*

PROOF. If a vector $\mathbf{v} \in \mathbb{K}[x]^{1 \times n}$ in the row space of \mathbf{M} is such that $\boldsymbol{\pi}(\mathbf{v}) \in J$, then $\boldsymbol{\pi}(\mathbf{v}) = \phi_J(\boldsymbol{\pi}(\mathbf{v}_{*,J}))$. This implies $(\boldsymbol{\pi}(\mathbf{M}) \cap J) \subseteq \phi_J(\boldsymbol{\pi}(\mathbf{M}_{*,J}))$ since the pivot index of any vector in the row space of \mathbf{M} (resp. $\mathbf{M}_{*,J}$) appears in $\boldsymbol{\pi}(\mathbf{M})$ (resp. $\boldsymbol{\pi}(\mathbf{M}_{*,J})$), see Section 2.4. It also immediately implies the equality whenever $\boldsymbol{\pi}(\mathbf{M}) \subseteq J$. \square

These properties lead to a fast method for computing the pivot support when $n \gg m$, relying on a black box `PIVOTSUPPORT` which efficiently finds the pivot support when $n \in O(m)$: one first considers the $2m$ left columns $\mathbf{M}_{*,\{1, \dots, 2m\}}$ and uses `PIVOTSUPPORT` to compute their pivot support $\boldsymbol{\pi}_1$. Then, Lemma 4.5 suggests to discard all columns of \mathbf{M} in $\{1, \dots, 2m\} \setminus \boldsymbol{\pi}_1$, thus obtaining a matrix \mathbf{M}_1 . Then, we repeat the same process to obtain $\mathbf{M}_2, \mathbf{M}_3$, etc.

Algorithm 3: WIDEMATRIXPIVOTSUPPORT

Input: matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$.

Output: the pivot support $\boldsymbol{\pi}(\mathbf{M})$ of \mathbf{M} .

Assumption: the algorithm `PIVOTSUPPORT` takes as input \mathbf{M} and returns $\boldsymbol{\pi}(\mathbf{M})$.

1. If $n \leq 2m$ then return `PIVOTSUPPORT`(\mathbf{M})
2. $\boldsymbol{\pi}_0 \leftarrow \text{PIVOTSUPPORT}(\mathbf{M}_{*,\{1, \dots, 2m\}})$
3. $\hat{\mathbf{M}} \leftarrow [\mathbf{M}_{*,\boldsymbol{\pi}_0} \quad \mathbf{M}_{*,\{2m+1, \dots, n\}}]$
4. $[\boldsymbol{\pi}_1 \quad \boldsymbol{\pi}_2] \leftarrow \text{WIDEMATRIXPIVOTSUPPORT}(\hat{\mathbf{M}})$,
such that $\max(\boldsymbol{\pi}_1) \leq \#\boldsymbol{\pi}_0$ and $\min(\boldsymbol{\pi}_2) > \#\boldsymbol{\pi}_0$.
5. Return $[\phi_{\boldsymbol{\pi}_0}(\boldsymbol{\pi}_1) \quad \phi_{\{2m+1, \dots, n\}}(\boldsymbol{\pi}_2)]$

PROPOSITION 4.6. *Algorithm 3 is correct. It uses at most $\lceil n/m \rceil$ calls to `PIVOTSUPPORT`, each with a $m \times k$ submatrix of \mathbf{M} as input, where $k \leq 2m$. If $m \leq n$ and `PIVOTSUPPORT` is Algorithm 2, then Algorithm 3 uses $O(m^{\omega-1}n \deg(\mathbf{M}))$ operations in \mathbb{K} .*

PROOF. The correctness follows from Lemma 4.5, and the operation count is obvious. If using Algorithm 2 for `PIVOTSUPPORT`, the correctness and cost bound follow from Proposition 4.4. \square

5 PRELIMINARIES ON SHIFTED FORMS

5.1 Shifted forms

The notions of reduced and Popov forms presented in Sections 2.3 and 2.4 can be extended by introducing additive integer weights in the degree measure for vectors, following [24, Sec. 3]: a *shift* is a tuple $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{Z}^n$, and the *shifted degree* of a row vector $\mathbf{p} = [p_1 \cdots p_n] \in \mathbb{K}[x]^{1 \times n}$ is

$$\text{rdeg}_s(\mathbf{p}) = \max(\deg(p_1) + s_1, \dots, \deg(p_n) + s_n) = \text{rdeg}(\mathbf{p}\mathbf{x}^{\mathbf{s}}),$$

where $\mathbf{x}^{\mathbf{s}} = \text{diag}(x^{s_1}, \dots, x^{s_n})$. Note that here $\mathbf{p}\mathbf{x}^{\mathbf{s}}$ may be over the ring of Laurent polynomials if $\min(\mathbf{s}) < 0$; below, actual computations will always remain over $\mathbb{K}[x]$. Note that with $\mathbf{s} = \mathbf{0}$ we recover the notion of degree used in the previous sections.

This leads to shifted reduced forms for cases where one is interested in matrices whose rows minimize the \mathbf{s} -degree, instead of the usual $\mathbf{0}$ -degree. The generalized definitions from Section 2 can be concisely described as follows. For a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$, its \mathbf{s} -row degree is $\text{rdeg}_s(\mathbf{M}) = \text{rdeg}(\mathbf{M}\mathbf{x}^{\mathbf{s}})$. If \mathbf{M} has no zero row, its \mathbf{s} -leading matrix is $\text{lm}_s(\mathbf{M}) = \text{lm}(\mathbf{M}\mathbf{x}^{\mathbf{s}})$, and the \mathbf{s} -pivot index and entries of \mathbf{M} are the pivot index and entries of $\mathbf{M}\mathbf{x}^{\mathbf{s}}$. The \mathbf{s} -pivot degree of \mathbf{M} is the tuple of the degrees of its \mathbf{s} -pivot entries; this is equal to $\text{rdeg}_s(\mathbf{M}) - \mathbf{s}_J$, where J is the \mathbf{s} -pivot index of \mathbf{M} and \mathbf{s}_J the corresponding subshift.

If \mathbf{M} has no zero row and $m \leq n$, then \mathbf{M} is in \mathbf{s} -reduced, \mathbf{s} -(ordered) weak Popov or \mathbf{s} -Popov form if $\mathbf{M}\mathbf{x}^{\mathbf{s}}$ has the respective non-shifted form, whenever $\min(\mathbf{s}) \geq 0$. Since adding a constant to all the entries of \mathbf{s} simply shifts the \mathbf{s} -degree of vectors by this constant, this does not change the \mathbf{s} -leading matrix or the \mathbf{s} -pivots, and thus does not affect the shifted forms. Therefore we can extend the definitions of these to also cover \mathbf{s} with negative entries; one may alternatively assume $\min(\mathbf{s}) = 0$ without loss of generality.

The \mathbf{s} -Popov form \mathbf{P} of a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ is the unique row basis of \mathbf{M} which is in \mathbf{s} -Popov form. The \mathbf{s} -pivot support of \mathbf{M} is the \mathbf{s} -pivot index of \mathbf{P} and is denoted by $\boldsymbol{\pi}_s(\mathbf{M}) \in \mathbb{Z}_{>0}^r$, where r is the rank of \mathbf{M} . For more details on shifted forms, we refer to [3].

Computationally, it is folklore that finding the shifted Popov form easily reduces to the non-shifted case: given a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$ and a nonnegative shift $\mathbf{s} \in \mathbb{Z}^n$, the non-shifted Popov form $\hat{\mathbf{P}}$ of $\mathbf{M}\mathbf{x}^{\mathbf{s}}$ has the form $\hat{\mathbf{P}} = \mathbf{P}\mathbf{x}^{\mathbf{s}}$, with \mathbf{P} the \mathbf{s} -Popov form of \mathbf{M} . If $m < n$ and the computation of $\hat{\mathbf{P}}$ can be carried out in $O(m^{\omega-1}n \deg(\mathbf{M}))$ operations, this approach yields \mathbf{P} in $O(m^{\omega-1}n(\deg(\mathbf{M}) + \text{amp}(\mathbf{s})))$. While this cost is satisfactory whenever $\text{amp}(\mathbf{s}) = O(\deg(\mathbf{M}))$, one may hope for improvements especially when $\text{amp}(\mathbf{s}) > m \deg(\mathbf{M})$. Indeed, Eq. (5) in Lemma 5.1 shows $\deg(\mathbf{P}) \leq m \deg(\mathbf{M})$, suggesting the target cost $O(m^{\omega}n \deg(\mathbf{M}))$ for the computation of \mathbf{P} .

5.2 Hermite form

A matrix $\mathbf{H} = [h_{i,j}] \in \mathbb{K}[x]^{r \times n}$ with $r \leq n$ is in *Hermite form* [11, 15, 20] if there are indices $1 \leq j_1 < \dots < j_r \leq n$ such that:

- $h_{i,j} = 0$ for $1 \leq j < j_i$ and $1 \leq i \leq r$,
- h_{i,j_i} is monic (therefore nonzero) for $1 \leq i \leq r$,
- $\deg(h_{i',j_i}) < \deg(h_{i,j_i})$ for $1 \leq i' < i \leq r$.

We call (j_1, \dots, j_r) the *Hermite pivot index* of \mathbf{H} ; note that it is precisely the column rank profile of \mathbf{H} .

For a matrix $\mathbf{M} \in \mathbb{K}[x]^{m \times n}$, its Hermite form $\mathbf{H} \in \mathbb{K}[x]^{r \times n}$ is the unique row basis of \mathbf{M} which is in Hermite form. We call *Hermite pivot support* of \mathbf{M} the Hermite pivot index of \mathbf{H} . Note that this is also the column rank profile of \mathbf{M} , since \mathbf{M} is unimodularly equivalent to \mathbf{H} (up to padding \mathbf{H} with zero rows).

For a given \mathbf{M} , the Hermite form can be seen as a specific shifted Popov form: defining the shift $\mathbf{h} = (nt, \dots, 2t, t)$ for any $t > \deg(\mathbf{H})$, the \mathbf{h} -Popov form of \mathbf{M} coincides with its Hermite form [3, Lem. 2.6]. Besides, the \mathbf{h} -pivot index of \mathbf{H} is (j_1, \dots, j_r) ; in other words, the Hermite pivot support $\boldsymbol{\pi}_h(\mathbf{M})$ is the column rank profile of \mathbf{M} .

5.3 Degree bounds for shifted Popov forms

The next result states that the unimodular transformation U between M and its s -Popov form P only depends on the submatrices of M and P formed by the columns in the s -pivot support. It also gives useful degree bounds for the matrices U and P ; for a more general study of such bounds, we refer to [3, Sec. 5].

LEMMA 5.1. *Let $M \in \mathbb{K}[x]^{m \times n}$ have full rank with $m \leq n$, let $s \in \mathbb{Z}^n$, let $P \in \mathbb{K}[x]^{m \times n}$ be the s -Popov form of M , and let $\pi = \pi_s(M)$ be the s -pivot index of P . Then $M_{*,\pi} \in \mathbb{K}[x]^{m \times m}$ is nonsingular, $P_{*,\pi}$ is its s_π -Popov form, and $U = P_{*,\pi} M_{*,\pi}^{-1} \in \mathbb{K}[x]^{m \times m}$ is the unique unimodular matrix such that $UM = P$.*

Furthermore, we have the following degree bounds:

$$\deg(P) \leq \deg(M) + \text{amp}(s), \quad (2)$$

$$\text{cdeg}(U_{*,i}) \leq |\text{rdeg}(M)| - \text{rdeg}(M_{i,*}) \quad \text{for } 1 \leq i \leq m, \quad (3)$$

$$\deg U \leq |\text{cdeg}(M_{*,\pi})|, \quad (4)$$

$$\deg(P) \leq \min(|\text{rdeg}(M)|, |\text{cdeg}(M')|) \leq m \deg(M) \quad \text{where } M' \text{ is } M \text{ with its zero columns removed.} \quad (5)$$

PROOF. Let $\hat{P} = M_{*,\pi}$, $\hat{M} = M_{*,\pi}$, and $\hat{s} = s_\pi$. Note first that \hat{P} is nonsingular and in \hat{s} -Popov form. Let $V \in \mathbb{K}[x]^{m \times m}$ be any unimodular matrix such that $VM = P$. Then in particular $V\hat{M} = \hat{P}$, hence \hat{M} is nonsingular and unimodularly equivalent to \hat{P} , which is therefore the \hat{s} -Popov form of \hat{M} . Besides, we have $V = \hat{P}\hat{M}^{-1} = U$.

It remains to prove the degree bounds. The first one comes from the minimality of P . Indeed, since P is an s -reduced form of M we have $\max(\text{rdeg}_s(P)) \leq \max(\text{rdeg}_s(M))$; the left-hand side of this inequality is at least $\deg(P) + \min(s)$ while its right-hand side is at most $\deg(M) + \max(s)$.

Let $\delta \in \mathbb{Z}_{\geq 0}^m$ be the s -pivot degree of P . Then, \hat{P} is in $(-\delta)$ -Popov form with $\text{rdeg}_{-\delta}(\hat{P}) = \mathbf{0}$ and $\text{cdeg}(\hat{P}) = \delta$ [12, Lem. 4.1]. Besides, \hat{P} is column reduced and thus $|\text{cdeg}(\hat{P})| = \deg(\det(\hat{P}))$ [13, Sec. 6.3.2], hence $|\delta| = \deg(\det(\hat{M}))$.

Let $t = (t_1, \dots, t_m) = \text{rdeg}(U^{-1})$. We obtain $\text{rdeg}_{-\delta}(\hat{M}) = \text{rdeg}_{-\delta}(U^{-1}\hat{P}) = \text{rdeg}_0(U^{-1}) = t$ by the predictable degree property (with shifts, see e.g. [26, Lem. 2.17]). Now, U being the transpose of the matrix of cofactors of U^{-1} divided by the constant $\det(U^{-1}) \in \mathbb{K} \setminus \{0\}$, we obtain $\text{cdeg}(U_{*,i}) \leq |t| - t_i$ for $1 \leq i \leq m$. Since $-\delta \leq \mathbf{0}$ we have $t = \text{rdeg}_{-\delta}(\hat{M}) \leq \text{rdeg}(M)$, hence $|t| - t_i \leq |\text{rdeg}(M)| - \text{rdeg}(M_{i,*})$. This proves (3).

Every entry of the adjugate of \hat{M} has degree at most $|\text{cdeg}(\hat{M})|$. Then, $U = \hat{P}\hat{M}^{-1}$ gives $\deg(U) \leq \deg(\hat{P}) - \deg(\det(\hat{M})) + |\text{cdeg}(\hat{M})|$. This yields (4) since $\deg(\hat{P}) = \max(\delta) \leq |\delta| = \deg(\det(\hat{M}))$.

The second inequality in (5) is implied by $|\text{rdeg}(M)| \leq m \deg(M)$. Besides, from $P = UM = \sum_{i=1}^m U_{*,i} M_{i,*}$ we see that (3) implies $\deg(P) \leq |\text{rdeg}(M)|$. For $j \in \pi$ we have $\text{cdeg}(P_{*,j}) \leq |\text{cdeg}(\hat{P})| = \deg(\det(\hat{M})) \leq |\text{cdeg}(M')|$. Now, let $j \in \{1, \dots, n\} \setminus \pi$: if $M_{*,j} = \mathbf{0}$ then $P_{*,j} = \mathbf{0}$, and otherwise it follows from (4) that $\text{cdeg}(P_{*,j}) = \deg(UM_{*,j}) \leq |\text{cdeg}(\hat{M})| + \text{cdeg}(M_{*,j}) \leq |\text{cdeg}(M')|$. \square

6 SHIFTED POPOV FORM WHEN THE PIVOT SUPPORT IS KNOWN

Now, we focus on computing the s -Popov form P of M when the s -pivot support $\pi_s(M)$ is known; here, M has full rank with $m < n$.

To exploit the knowledge of $\pi = \pi_s(M)$, a first approach follows from Remark 1: use Algorithm 1 with L such that $L_{*,\{1,\dots,n\} \setminus \pi}$ is the identity matrix and its other columns are zero. Then, it is easily checked that $C = Lx^{\max(\text{rdeg}_s(M)) - s}$ is a completion of $\hat{M} = Mx^s$; hence Algorithm 1 returns the Popov form $\hat{P} = Px^s$ of \hat{M} . This yields P deterministically in $O(n^\omega(\deg(M) + \text{amp}(s)))$ operations.

Both factors in this cost bound are unsatisfactory in some parameter ranges. When $n \gg m$, a sensible improvement would be to replace the matrix dimension factor n^ω with one which has the exponent on the smallest dimension, such as $m^{\omega-1}n$. Similarly, when $\text{amp}(s) \gg m \deg(M)$, a sensible improvement would be to replace the polynomial degree factor $\deg(M) + \text{amp}(s)$ with one suggested by the bounds on $\deg(P)$ given in Eq. (5) of Lemma 5.1.

We achieve both improvements with our second approach, which works in three steps and is formalised as Algorithm 4. First, we compute the s_π -Popov form of the submatrix $M_{*,\pi}$, which can be done efficiently since this submatrix is square and nonsingular. Then, we use polynomial matrix division to obtain the unimodular transformation $U \in \mathbb{K}[x]^{m \times m}$ such that $M_{*,\pi_s(M)} = UP_{*,\pi_s(M)}$. Lastly, we compute the remaining part of the s -Popov form of M as $U^{-1}M_{*,\{1,\dots,n\} \setminus \pi}$. Note that, even for $s = \mathbf{0}$, all entries of U^{-1} may have degree in $\Theta(m \deg(M))$; we avoid handling such large degrees by computing this product truncated at precision x^δ , where δ is a (strict) upper bound on the degree of the s -Popov form P . For example, if $s = \mathbf{0}$ we can take $\delta = 1 + \deg(M)$.

Algorithm 4: KNOWN SUPPORT POPOV

Input:

- matrix $M \in \mathbb{K}[x]^{m \times n}$ with full rank and $m < n$,
- shift $s \in \mathbb{Z}^n$,
- the s -pivot support $\pi = \pi_s(M)$ of M ,
- bound $\delta \in \mathbb{Z}_{>0}$ on the degree of the s -Popov form of M .

Default: $\delta = 1 + \min(|\text{rdeg}(M)|, |\text{cdeg}(M')|, \deg(M) + \text{amp}(s))$, where M' is M with zero columns removed.

Output: the s -Popov form of M .

1. $P \leftarrow$ zero matrix in $\mathbb{K}[x]^{m \times n}$
2. $P_{*,\pi} \leftarrow \text{NONSINGULARPOPOV}(M_{*,\pi}, s_\pi)$
3. $U \leftarrow M_{*,\pi} P_{*,\pi}^{-1} \in \mathbb{K}[x]^{m \times m}$
4. $\delta \leftarrow \min(\delta, 1 + \max(\text{rdeg}_{s_\pi}(P_{*,\pi})) - \min(s_{\{1,\dots,n\} \setminus \pi}))$
5. $P_{*,\{1,\dots,n\} \setminus \pi} \leftarrow U^{-1} M_{*,\{1,\dots,n\} \setminus \pi} \bmod x^\delta$
6. *Return* P

PROPOSITION 6.1. *Algorithm 4 is correct and uses $O^*(m^{\omega-1}n\delta)$ operations in \mathbb{K} , where*

$$\delta = 1 + \min(|\text{rdeg}(M)|, |\text{cdeg}(M')|, \deg(M) + \text{amp}(s)),$$

and M' is M with zero columns removed.

PROOF. Let $Q \in \mathbb{K}[x]^{m \times n}$ be the s -Popov form of M . For correctness we prove that $P = Q$. The first part of Lemma 5.1 shows that indeed $Q_{*,\pi} = P_{*,\pi}$, and that $U = M_{*,\pi} P_{*,\pi}^{-1} = M_{*,\pi} Q_{*,\pi}^{-1}$ computed at Step 3 is the unimodular matrix such that $M = UQ$.

The last item of Lemma 5.1 proves that the input default value of δ is more than $\deg(Q)$. Besides, by definition of s -pivots and s -Popov form, the column j of Q has degree at most

$$\max(\text{rdeg}_{s_\pi}(Q_{*,\pi})) - s_j = \max(\text{rdeg}_{s_\pi}(P_{*,\pi})) - s_j.$$

It follows that $\delta > \deg(\mathbf{Q}_{*,\{1,\dots,n\}\setminus\pi})$ holds after Step 4, and thus the submatrix $\mathbf{Q}_{*,\{1,\dots,n\}\setminus\pi}$ is equal to the truncated product $\mathbf{U}^{-1} \mathbf{M}_{*,\{1,\dots,n\}\setminus\pi} \bmod x^\delta$ computed at Step 5. Hence $\mathbf{Q} = \mathbf{P}$.

Now we explain the cost bound. Step 2 uses $O(m^\omega \deg(\mathbf{M}_{*,\pi}))$ operations, by Theorem 2.2. Step 3 has the same cost by Lemma 6.2 below; note that $\mathbf{P}_{*,\pi}$ is in s_π -Popov form and thus column reduced. This is within the announced bound since

$$O(m^\omega \deg(\mathbf{M}_{*,\pi})) \subseteq O(m^{\omega-1} n \deg(\mathbf{M}))$$

and $\deg(\mathbf{M}) \leq \delta$ holds by definition of δ .

Finally, Step 5 costs $O(m^{\omega-1} n \delta)$ operations in \mathbb{K} : since $\mathbf{U}(0) \in \mathbb{K}^{m \times m}$ is invertible, the truncated inverse of \mathbf{U} is computed by Newton iteration in time $O(m^\omega \delta)$; then, the truncated product uses $O(m^\omega [(n-m)/m] \delta)$ operations. \square

At Step 3, we compute a product of the form $\mathbf{B}\mathbf{A}^{-1}$, knowing that it has polynomial entries and that \mathbf{A} is column reduced; in particular, $\deg(\mathbf{B}\mathbf{A}^{-1}) \leq \deg(\mathbf{B})$ [19, Lem. 3.1]. Then, it is customary to obtain $\mathbf{B}\mathbf{A}^{-1}$ via a Newton iteration on the “reversed matrices” (see e.g. [22, Chap. 5] and [26, Chap. 10]).

LEMMA 6.2. *For a column reduced matrix $\mathbf{A} \in \mathbb{K}[x]^{m \times m}$ and a matrix $\mathbf{B} \in \mathbb{K}[x]^{m \times m}$ which is a left multiple of \mathbf{A} , the quotient $\mathbf{B}\mathbf{A}^{-1}$ can be computed using $O(m^\omega \deg(\mathbf{B}))$ operations in \mathbb{K} .*

PROOF. We follow Steps 1 and 2 of the algorithm PM-QUOREM from [19], on input \mathbf{A} , \mathbf{B} , and $d = \deg(\mathbf{B}) + 1$; hence the requirement $\text{cdeg}(\mathbf{B}) < \text{cdeg}(\mathbf{A}) + (d, \dots, d)$ is satisfied. It is proved in [19, Prop. 3.4] that these steps correctly compute the quotient $\mathbf{B}\mathbf{A}^{-1}$; yet we do a different cost analysis since the assumptions on parameters in [19, Prop. 3.4] might not be satisfied here.

Step 1 of PM-QUOREM computes a type of reversals $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ of the matrices \mathbf{A} and \mathbf{B} : this uses no arithmetic operation. These matrices also have dimensions $m \times m$ and the constant coefficient of $\hat{\mathbf{A}}$ is invertible because \mathbf{A} is column reduced. Step 2 computes the truncated product $\hat{\mathbf{B}}\hat{\mathbf{A}}^{-1} \bmod x^{d+1}$, which can be done via Newton iteration in $O(m^\omega d)$ operations in \mathbb{K} . \square

Since Algorithm 4 works for an arbitrary shift, it allows us in particular to find the Hermite form of \mathbf{M} when its Hermite pivot support is known. It turns out that the latter can be computed efficiently via a column rank profile algorithm from [26].

PROOF OF THEOREM 1.2. Here, the integer δ is defined as

$$\delta = 1 + \min(|\text{rdeg}(\mathbf{M})|, |\text{cdeg}(\mathbf{M}')|), \quad (6)$$

where \mathbf{M}' is \mathbf{M} with zero columns removed.

Let $\mathbf{h} = (n\delta, \dots, 2\delta, \delta)$. By Lemma 5.1, δ is more than the degree of the Hermite form of \mathbf{M} ; therefore the \mathbf{h} -Popov form of \mathbf{M} is also its Hermite form (see Section 5.2). Thus, up to the knowledge of the Hermite pivot support $\pi_{\mathbf{h}}(\mathbf{M})$ of \mathbf{M} , we can compute the Hermite form of \mathbf{M} using $O(m^{\omega-1} n \delta)$ operations via Algorithm 4.

As mentioned in Section 5.2, $\pi_{\mathbf{h}}(\mathbf{M})$ is also the column rank profile of \mathbf{M} . It is shown in [26, Sec. 11.2] how to use row basis and kernel basis computations to obtain this rank profile in $O(m^{\omega-1} n \sigma)$ operations, where $\sigma = \lceil |\text{rdeg}(\mathbf{M})|/m \rceil$ is roughly the average row degree of \mathbf{M} . We have $\sigma \leq 1 + |\text{rdeg}(\mathbf{M})|$ by definition, and it is easily verified that $|\text{rdeg}(\mathbf{M})|/m \leq |\text{cdeg}(\mathbf{M}')|$, hence $\sigma \leq \delta$. \square

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