



**HAL**  
open science

# Convex Optimisation Method for Matching Filters Synthesis

David Martínez Martínez, Gibin Bose, Fabien Seyfert, Martine Olivi

► **To cite this version:**

David Martínez Martínez, Gibin Bose, Fabien Seyfert, Martine Olivi. Convex Optimisation Method for Matching Filters Synthesis. IWWMF 2018 - 7th International Workshop on Microwave Filters, ESA; CNES, Apr 2018, Noordwijk, Netherlands. hal-01778631

**HAL Id: hal-01778631**

**<https://unilim.hal.science/hal-01778631>**

Submitted on 1 Jun 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# CONVEX OPTIMISATION METHOD FOR MATCHING FILTERS SYNTHESIS

David Martínez Martínez<sup>1,2</sup>, Gibin Bose<sup>1</sup>, Fabien Seyfert<sup>1</sup>, and Martine Olivi<sup>1</sup>

<sup>1</sup>*INRIA Sophia Antipolis Méditerranée*  
2004 Route des Lucioles - 06902 Valbonne (France)  
david.martinez@inria.fr

<sup>2</sup>*XLIM Université de Limoges*  
123, avenue Albert Thomas - 87060 Limoges (France)

## Abstract

In this paper, a certified synthesis technique is presented for the design of matching filters that combines convex optimisation with the Fano-Youla matching theory. This technique is applicable to any rational load and provides lower hard bounds for the best matching level, as well as a practical synthesis of a matching filter approaching those bounds. Furthermore, if the load is a rational function of degree 1, the optimal matching filter is synthesized, yielding in this case an extension of the classical filter synthesis for resistive loads. As example, a dual-band matching filter is conceived for a dual-band antenna. Additionally a single-band filter is implemented in SIW technology to match a single-band antenna.

## 1. INTRODUCTION

There exists a remarkable literature in the field of broadband matching. In [1–3] broad-band matching was first introduced based on the use of the Darlington two port equivalent and extraction procedures. The theory was first reviewed in [1] where the problem of matching an  $RC$ -load is considered as the design of a lowpass filtering network where an  $RC$ -element is fixed. In [2] this problem was extended to the case of a generic load by using the *Darlington equivalent* and reformulated in [3] as a complex interpolation problem. The theory was, for example, used to synthesize matching networks with a Tchebychev type power gain transducer [4], nevertheless this type of responses are known to be non optimal in terms of matching performances unless the load is a constant impedance. This approach was therefore progressively replaced by the optimization based *real frequency* technique of Carlin [5] which is more oriented to practical applications. Additionally in [6] the matching problem was solved optimally by considering the broader class of infinite dimension functions  $H^\infty$  and therefore providing hard bounds for the matching problem in finite dimension.

In this work we use the Fano-Youla matching theory combined with convex optimisation to formulate the matching problem. Within this framework, we introduce in section 2 a convex relaxation of the generalised matching problem available in the literature providing hard lower bounds for the original problem when rational filters of finite degree are considered. In section 3 we show an example of matching filter synthesis for a dual-band antenna. Finally, in section 3.1 a practical example is presented to validate the proposed algorithm.

## 2. THEORY

The matching problem aims to minimise the reflection of the power transmitted to a given load within a specified frequency band. The load is represented as a 2-port device ( $A$ ) in Fig. 1. Usually the power is transmitted to the load through a filter ( $F$ ) that rejects out of band signals. Both devices, the filter connected to the load compose the global system ( $S$ ). It is important to specify that if only the input reflection of the load  $A_{11}$  is known, a *Darlington equivalent* of the load (see [7]) yields a loss-less two port network with the same input reflection  $A_{11}$ . Following the Fano-Youla approach to the matching problem, the system  $S$  is conceived first, followed by the de-embedding of the load.

Let us introduce first some notations and definitions. Consider the complex variable  $\lambda = \omega + j\sigma$  where  $\omega$  is the frequency variable. We denote by  $\mathbb{C}^+$  the open upper half plane,  $\mathbb{C}^+ = \{\lambda : \Im(\lambda) > 0\}$  and by  $\mathbb{C}^-$

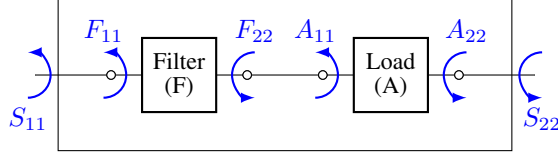


Figure 1: System ( $S = F \circ A$ ) and reflection coefficients.

the open lower half plane;  $\overline{\mathbb{C}^-}$  denotes the closed lower half plane  $\overline{\mathbb{C}^-} = \{\lambda : \Im(\lambda) \leq 0\}$ . In this work we consider  $\mathbb{C}^-$  as the analyticity domain.

**Definition 2.1** (Scattering matrix). *We call scattering matrix a rational  $2 \times 2$  matrix of the complex variable  $\lambda$ , unitary for  $\lambda \in \mathbb{R}$  and analytic in  $\mathbb{C}^-$ . Its elements are scalar rational functions contractive in  $\mathbb{C}^-$ , namely Schur functions.*

Consider the scattering matrices  $S$ ,  $F$  and  $A$  represented in the Belevitch form [8]

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \frac{1}{q} \begin{pmatrix} \epsilon p^* & -\epsilon r^* \\ r & p \end{pmatrix} \quad (1)$$

with  $\epsilon$  a uni-modular constant, and  $q, p, r$  polynomials satisfying  $qq^* = pp^* + rr^*$  with  $p^*(\lambda) = \overline{p(\overline{\lambda})}$ . Note that  $q$  is a *stable* polynomial, that is, with all roots in  $\mathbb{C}^+$ .

**Definition 2.2** (Chaining). *We define the chaining  $F_{22} \circ A$  of a Schur function  $F_{22}$  and a matrix  $A$  in the form (1) as the output reflection of the global system  $S_{22}$  composed of the cascade of  $F$  and  $A$  (see Fig. 1).*

$$S_{22} = F_{22} \circ A = A_{22} + \frac{A_{21}F_{22}A_{12}}{1 - A_{11}F_{22}} \quad (2)$$

**Definition 2.3** (Feasibility). *We define a function  $S_{22}$  as feasible for a given load  $A$  if there exists a Schur function  $F_{22}$ , such that  $F_{22} \circ A = S_{22}$ . Additionally we denote by  $\mathbb{F}$  the set of feasible functions  $S_{22}$  for a given load  $A$ .*

Note that  $\mathbb{F}$  is the image of the set of Schur functions under the application  $f \rightarrow f \circ A$ . If  $S_{22}$  is admissible for a load  $A$ , then the function  $F_{22}$  such that  $F_{22} \circ A = S_{22}$  expresses as:

$$F_{22} = \frac{A_{22} - S_{22}}{\det A - A_{11}S_{22}} \quad (3)$$

Next we present a characterisation of  $\mathbb{F}$  by a set of interpolation conditions at the transmission zeros of  $A$ .

**Definition 2.4** (Transmission zeros). *We define the transmission zeros associated to a matrix function  $S$  in the form (1) as the zeros in  $\mathbb{C}^-$  (possibly at  $\infty$ ) of  $S_{12}S_{21}(\lambda)$ :*

$$\text{tz}[S] = \left\{ \lambda \in \overline{\mathbb{C}^-} : S_{12}S_{21}(\lambda) = 0 \right\} \quad (4)$$

where we consider the classical multiplicity of the transmission zeros in  $\mathbb{C}^-$  and half of the multiplicity for the transmission zeros in  $\mathbb{R}$ . Also remark that the transmission zeros, being in  $\overline{\mathbb{C}^-}$ , cannot simplify with the zeros of  $q$ .

Note that, if  $S_{12}$  is assumed to be *minimum phase*<sup>1</sup> (i.e. has no zeros in  $\mathbb{C}^-$ ), then, the finite transmission zeros of  $S_{22}$  are the zeros of  $r$ . In that case  $r$  is uniquely determined by spectral factorisation of the positive polynomial  $R = qq^* - pp^*$  and therefore the matrix  $S$  is recovered from the polynomials  $p, q$ , up to the uni-modular constant  $\epsilon$ .

A core result of Fano's-Youla's matching theory is the necessary and sufficient conditions for  $F_{22}$  to be Schur in  $\mathbb{C}^-$ . These conditions represent the characterisation of the set  $\mathbb{F}$ .

**Proposition 2.5** (Characterisation of  $\mathbb{F}$ ). *Consider a lossless load  $A$  with transmission zeros  $\alpha_i$ , and  $\mathbb{F}$  its feasible set. A rational Schur function  $S_{22}$  belongs to  $\mathbb{F}$  iff<sup>2</sup>*

<sup>1</sup>Minimum phase functions, also called *outer*, have many useful properties for our purpose, see e.g. [9, Th.4.6]

<sup>2</sup>Equivalent forms of (5b) and (5c) can be used if the transmission zeros  $\alpha_i$  occurs at  $\lambda = \infty$ .

- At each transmission zero  $\alpha_i$  of multiplicity  $m_i$  of  $A$ , the following interpolation conditions hold<sup>3</sup>:

$$\left( D^k S_{22} \right) [\alpha_i] = \xi_{i,k} \quad 0 \leq k \leq m_i - 1 \quad \forall \alpha_i \in \mathbb{C}^- \quad (5a)$$

$$\left( D^k S_{22} \right) [\alpha_i] = \xi_{i,k} \quad 0 \leq k \leq 2m_i - 2 \quad \forall \alpha_i \in \mathbb{R} \quad (5b)$$

$$\left( D^k j \ln S_{22} \right) [\alpha_i] \leq \psi_{i,2m_i-1} \quad \forall \alpha_i \in \mathbb{R} \quad (5c)$$

with  $\xi_{i,k} = (D^k A_{22}) [\alpha_i]$  and  $\psi_{i,k} = (D^k j \ln A_{22}) [\alpha_i]$ .<sup>4</sup>

## 2.1. A Convex Relaxation of the Matching Problem

With these definitions, we can state the general form of the matching problem. Notice that in [6] the matching problem is stated as the minimisation of the reflection level without any additional constraints on  $S_{22} \in \mathbb{F}$ . It is only supposed that  $S_{22}$  belong to the infinite dimensional class of functions  $H^\infty$ . In this work however, we constrain  $S_{22} \in \mathbb{F}$  to be rational in the form (1) with  $p, r \in \mathbb{P}^N$  (the set of polynomials of degree  $N$ ). Additionally we suppose that the polynomial  $r$  is fixed as it is customary in classical filter synthesis. Furthermore we assume that the transmission zeros  $\alpha_i$  are also roots of  $r$ . Thus  $r$  will have roots at the transmission zeros  $\alpha_i$  as well as any other possible transmission zeros fixed in advance. Applying the change of variable:  $pp^* = P$ ,  $rr^* = R$  we denote with  $\mathbb{F}_R^N$  the set of rational functions  $S_{22} \in \mathbb{F}$  of degree  $N$  with the *transmission polynomial*  $R \in \mathbb{P}_+^{2N}$ .

$$\mathbb{F}_R^N = \left\{ S_{22} \in \mathbb{F} \mid \exists P \in \mathbb{P}_+^{2N} : |S_{22}(\omega)|^2 = \frac{P(\omega)}{P(\omega) + R(\omega)}; \forall \omega \in \mathbb{R} \right\} \quad (6)$$

where  $\mathbb{P}_+^{2N}$  denotes the set of positive polynomials of degree at most  $2N$ . Therefore  $S_{22}(P)$  is obtained as the minimum phase factor of  $|S_{22}(P)|^2 = (1 + \frac{R}{P})^{-1}$ . We state the problem as

**Problem (P).**

$$\begin{aligned} \text{Find:} \quad & l = \min_{S_{22} \in \mathbb{F}_R^N} \max_{\omega \in \mathbb{I}_1} |S_{22}(\omega)|^2 \\ \text{Subject to:} \quad & \gamma \leq |S_{22}(\omega)|^2 \quad \forall \omega \in \mathbb{I}_2 \end{aligned}$$

where  $\mathbb{I}_1$  represents the passband,  $\mathbb{I}_2$  the stopband and  $\gamma$  the desired rejection level in the interval  $\mathbb{I}_2$ .

We introduce now a convex relaxation of problem  $\mathcal{P}$  by considering the notion of *admissibility*.

**Definition 2.6** (Admissibility). *A minimum phase Schur function  $U$  is admissible for a load  $A$  iff there exists  $S_{22} \in \mathbb{F}$  such that for all  $\omega \in \mathbb{R}$ ,  $|S_{22}(\omega)| \leq |U(\omega)|$ . We denote by  $\mathbb{G}$  the set of admissible  $U$ .*

For every admissible  $U$  there exist  $S_{22} \in \mathbb{F}$  such that  $S_{22}U^{-1}$  is a Schur function. Thus  $\mathbb{G}$  can be characterised by reformulating (5a) to (5c) on  $B$  where  $B = S_{22}U^{-1}$ .

**Proposition 2.7** (Characterisation of  $\mathbb{G}$ ). *A minimum phase rational Schur reflection  $U$  is admissible for a load  $A$  with transmission zeros  $\alpha_i$  of multiplicity  $m_i$  if and only if*

- There exists a Schur function  $B$  satisfying (7a) to (7c) at every transmission zero  $\alpha_i$  of  $A$ .

$$\left( D^k B \right) [\alpha_i] = \Xi_{i,k} \quad 0 \leq k \leq m_i - 1 \quad \forall \alpha_i \in \mathbb{C}^- \quad (7a)$$

$$\left( D^k B \right) [\alpha_i] = \Xi_{i,k} \quad 0 \leq k \leq 2m_i - 2 \quad \forall \alpha_i \in \mathbb{R} \quad (7b)$$

$$\left( D^k j \ln B \right) [\alpha_i] \leq \Psi_{i,2m_i-1} \quad \forall \alpha_i \in \mathbb{R} \quad (7c)$$

where we denote  $\Xi_{i,k} = (D^k A_{22}U^{-1}) [\alpha_i]$  and also  $\Psi_{i,k} = (D^k j \ln A_{22}U^{-1}) [\alpha_i]$ .

<sup>3</sup>Note this condition does not require the transmission zeros of  $A$  to be present in the system  $S$  as long as the interpolation conditions are satisfied. Nevertheless if the transmission zeros of  $A$  are not present in  $S$ , the matching filter obtained after embedding will include those transmission zeros  $\alpha_i$  at the expense of not being of minimal degree. In this paper, we assume that the transmission zeros of the load  $A$  are also present in  $S$  with at least the same multiplicity, thus obtaining a matching filter  $F$  of minimal degree.

<sup>4</sup>The symbol  $D^k$  stands for the  $k$ -th derivative.

As before, we are interested in the admissible functions that are rational with a given transmission polynomial. Therefore, as it was done for  $\mathbb{F}$ , we define  $\mathbb{G}_R^N$  the set of functions  $S_{22} \in \mathbb{G}$  of minimum phase whose modulus square can be expressed in a rational form as  $|S_{22}(\omega)|^2 = \frac{P(\omega)}{P(\omega)+R(\omega)}$  with  $P, R \in \mathbb{P}_+^{2N}$ .

Now define  $U_P(\omega)$  as the outer spectral factor of  $(1 + R(\omega)/P(\omega))^{-1}$  with  $P, R \in \mathbb{P}_+^{2N}$ . At this point it is possible to find a convex parametrisation of  $U_P \in \mathbb{G}_R^N$  in function of the polynomials  $P \in \mathbb{P}_+^{2N}$ .

**Definition 2.8** (Admissible polynomials). *We denote by  $\mathbb{H}_R^N$  the set of  $P \in \mathbb{P}_+^{2N}$  such that  $U_P \in \mathbb{G}_R^N$ .*

**Proposition 2.9** (Convexity). *The set  $\mathbb{H}_R^N$  is a convex set.*

*Proof.* Suppose  $P_1, P_2 \in \mathbb{H}_R^N$ , then there exists  $S_1, S_2 \in \mathbb{F}$  such that  $\omega \in \mathbb{R}, |S_1(\omega)| \leq |U_{P_1}(\omega)|$  and  $|S_2(\omega)| \leq |U_{P_2}(\omega)|$ . We verify now that  $P_3 = \lambda P_1 + (1 - \lambda)P_2 \in \mathbb{H}_R^N$ . Notice  $|U_P| = \sqrt{(1 + \frac{R}{P})^{-1}}$  is concave in  $P$ . The concavity implies:

$$|U_{P_3}(\omega)| \geq \lambda |U_{P_1}(\omega)| + (1 - \lambda) |U_{P_2}(\omega)| \quad \forall \omega \in \mathbb{R} \quad (8)$$

The function  $S_3 = \lambda S_1 + (1 - \lambda)S_2$  satisfies (5a) to (5c). Therefore  $S_3 \in \mathbb{G}_R^N$ . From the triangle inequality:

$$|S_3(\omega)| \leq \lambda |S_1(\omega)| + (1 - \lambda) |S_2(\omega)| \quad \forall \omega \in \mathbb{R} \quad (9)$$

It follows from (9) and (8) that  $|S_3(\omega)| \leq |U_{P_3}(\omega)|$ . Thus  $U_{P_3} \in \mathbb{G}_R^N$  and therefore  $P_3 \in \mathbb{H}_R^N$ .  $\square$

**Remark 1.** *In problem  $\mathcal{P}$  the optimisation is done with respect of the rational functions  $S_{22}$  feasible for the given load ( $\mathbb{F}_R^N$ ). Note that  $\mathbb{F}_R^N$  is not a convex set. Therefore, in order to state a convex optimisation problem, we have defined a relaxed set  $\mathbb{G}_R^N$  of functions that are not necessarily feasible. However if a reflection  $g \in \mathbb{G}_R^N$  is not feasible for the load, then there must exist a feasible reflection  $f \in \mathbb{F}_R^N$  whose modulus is equal or better than the modulus of  $g$ , namely  $|f(\omega)| \leq |g(\omega)|, \forall \omega \in \mathbb{R}$ . Note that  $\mathbb{F}_R^N \subset \mathbb{G}_R^N$ . In this section we parametrise each function  $U_P \in \mathbb{G}_R^N$  by a positive polynomial  $P$ . This parametrisation leads to the set of positive polynomials  $P$  such that  $U_P \in \mathbb{G}_R^N$ , namely  $\mathbb{H}_R^N$ . Finally we prove that  $\mathbb{H}_R^N$  is a convex set.*

### 2.1.1. Generalised Matching Problem

We state next the matching problem with respect of the set of polynomials  $\mathbb{H}_R^N$  (a convex set). Note that the maximisation of the reflection level with respect to  $\omega$  is already convex. Thus we obtain a convex relaxation of problem  $\mathcal{P}$ .

**Problem ( $\mathcal{P}_C$ ).** *Find  $L = \min_{P \in \mathbb{H}_R^N} \max_{\omega \in \mathbb{I}_1} \frac{P}{R}(\omega)$*

**Proposition 2.10** (Convexity and unicity). *Problem  $\mathcal{P}_C$  is convex and admits a unique solution.*

For simplicity we consider here no rejection constraints. However, as it is known in classical filter synthesis, linear constraints on the modulus of  $U_P$  can be transformed to linear constraints on the filtering function  $P/R$  [10].

$$|U_P(\omega)|^2 = \frac{P(\omega)}{R(\omega) + P(\omega)} \geq \gamma \quad \iff \quad P(\omega) \geq \Gamma \cdot R(\omega) \quad \Gamma = (1/\gamma - 1)^{-1} \quad (10)$$

## 2.2. Practical Implementation of the Relaxed Problem $\mathcal{P}_C$

In order to assure the admissibility of  $U_P$  in problem  $\mathcal{P}_C$  it is necessary to guarantee the existence of  $B$  verifying (7a) to (7c). If we consider the simplest case, where the load has only simple transmission zeros in  $\mathbb{C}^-$ , the problem is equivalent to the classical *Nevanlinna-Pick interpolation* problem. The Nevanlinna-Pick theorem states [9]:

**Theorem 2.11** (Nevanlinna-Pick Interpolation Theorem). *Given  $\gamma_1, \gamma_2 \dots \gamma_n \in \overline{\mathbb{D}}$  and  $\alpha_1, \alpha_2 \dots \alpha_n \in \mathbb{C}^-$ . There exist a Schur function  $B : \mathbb{C}^- \rightarrow \overline{\mathbb{D}}$  satisfying  $B(\alpha_i) = \gamma_i$  if and only if the Pick matrix*

$$\Delta_{i,k} = j \left( \frac{1 - \gamma_i \overline{\gamma_k}}{\alpha_i - \overline{\alpha_k}} \right) \quad (11)$$

*is positive semi-definite. Furthermore,  $B$  is unique iff  $\Delta$  is singular. In this case  $B$  is a Blaschke product.*

The *Nevanlinna-Pick Interpolation Theorem* is generalised in [11] to consider interpolating points  $\alpha_i \in \mathbb{C}^-$  and interpolation conditions on the derivatives. The generalised *Nevanlinna-Pick Interpolation Theorem* states the necessary and sufficient condition for the existence of a Schur function  $B$  satisfying (7a) to (7c). Additionally *Nevanlinna's* theory also provides a parametrisation of all possible functions  $B$ . For simplicity consider a load with only simple transmission zeros  $\alpha_i \in \mathbb{C}^-$ .

**Proposition 2.12** (Admissibility condition). *Consider a load  $A$  with simple transmission zeros  $\alpha_i \in \mathbb{C}^-$  and the transmission polynomial  $R \in \mathbb{P}_+^{2N}$ . A polynomial  $P \in \mathbb{P}_+^{2N}$  is admissible iff  $\Delta(P) \succeq 0$  where*

$$\Delta(P)_{i,k} = j \left( \frac{1 - (A_{22}U_P^{-1})[\alpha_i] \cdot \overline{(A_{22}U_P^{-1})[\alpha_k]}}{\alpha_i - \overline{\alpha_k}} \right) \quad (12)$$

Considering a load  $A$  of degree  $M$  with simple transmission zeros  $\alpha_i \in \mathbb{C}^-$ , we can use the previous theorem to state  $\mathcal{P}_C$  as the minimisation of  $P/R$  on the passband over all  $P \in \mathbb{P}_+^{2N}$  under the condition  $\Delta(P) \succeq 0$  to ensure that  $P \in \mathbb{H}_R^N$ .

### 2.2.1. Bounds for the solution of problem $\mathcal{P}$

For each  $U_P \in \mathbb{G}$ , there exist a function  $B_P$  such that  $S_{22} = U_P \cdot B_P \in \mathbb{F}$ . Consider  $\widehat{P}$  the optimal  $P$  of  $\mathcal{P}_C$ . Denote  $\widehat{U} = U_{\widehat{P}}$ ,  $\widehat{B} = B_{\widehat{P}}$  and  $\widehat{S}_{22} = \widehat{U} \cdot \widehat{B}$ . Then it can be proved that

1. The degree of  $B_P$  equals the rank of  $\Delta(P)$ .
2. The matrix  $\Delta(\widehat{P})$  is singular. Therefore the unique function  $\widehat{B}$  verifying (7a) to (7c) is a *Blaschke* product of degree  $L \leq M - 1$ .

$$\widehat{B}(\omega) = \frac{\prod_{i=1}^L (\omega - \beta_i)}{\prod_{i=1}^L (\omega - \overline{\beta_i})} \quad (13)$$

3. The degree of  $\widehat{S}_{22}$  is bounded between  $N$  and  $N+M-1$ .

From a formal point of view, problem  $\mathcal{P}_C$  provides hard lower bounds for the attainable matching level in problem  $\mathcal{P}$  since  $\mathbb{F}_R^N \subset \mathbb{G}_R^N$ . Additionally we can construct a function  $\widehat{S}_{22} = \widehat{U} \cdot \widehat{B}$  that attains such bound. However this feasible function is not always in  $\mathbb{F}_R^N$  since some complex transmission zeros (as many as the degree of  $\widehat{B}$ ) may be required. Therefore  $\widehat{S}_{22} \in \mathbb{F}_{\widehat{R}}^N$  with  $\widehat{R} = R \cdot \prod_{i=1}^L (\omega - \beta_i)$ .

Additionally note that for a load of degree  $M = 1$ , the obtained Blaschke product is always of degree 0. In this case the relaxation made in problem  $\mathcal{P}_C$  is exact providing the optimal solution to problem  $\mathcal{P}$ .

### 2.2.2. Implementation as a Semi-definite Program

Problem  $\mathcal{P}_C$  can be solved optimally by non-linear semi-definite programming techniques. Indeed, the constraint on the positivity of  $P$  in  $\mathcal{P}_C$  can be recasted by means of *linear matrix inequalities* by imposing the positive semi-definiteness of a matrix  $A$  [12]. We obtain then a *semi-definite program* with one non-linear constraint  $\Delta(P) \succeq 0$  that ensures the admissibility of  $P$ . Those constraints are implemented by using a barrier/penalty function. Linear matrix inequalities are handled by the standard logarithmic barrier meanwhile the non-linear matrix inequality is ensured by the penalty function presented in [13]. Note that adding more passbands or some rejection constraints amounts to add some extra positive definite matrices to ensure the positivity of  $LR(\omega) - P(\omega)$  in the  $i^{\text{th}}$ -passband, or to guarantee that  $P(\omega) \geq \Gamma \cdot R(\omega)$  is satisfied in the  $j^{\text{th}}$ -stopband.

## 3. RESULTS

As proof of concept, we present a synthesis example for a GNSS receiver matching a dualband antenna in the GPS/GALILEO bands: L2 (1.21-1.24GHz), E6 (1.26-1.3GHz), L1 (1.55-1.6GHz). Fig. 2 shows a comparison between the reflection of the antenna ( $A_{11}$ ) and the results of  $\mathcal{P}_C$  ( $S_{22}$ ) by taking  $N = 7$ . By using the matching filter, the reflection at the right edge of the band E6 (1.3GHz) has been improved from  $-1.4\text{dB}$  to  $-7.95\text{dB}$  representing an improvement of 450%. Parameters  $F_{22}$  and  $F_{21}$  of the matching filter that provides this result are shown in Fig. 3 together with the load reflection  $A_{11}$ . Furthermore we show in Fig. 4 the bound for the optimal reflection level attainable in  $\mathcal{P}$ .

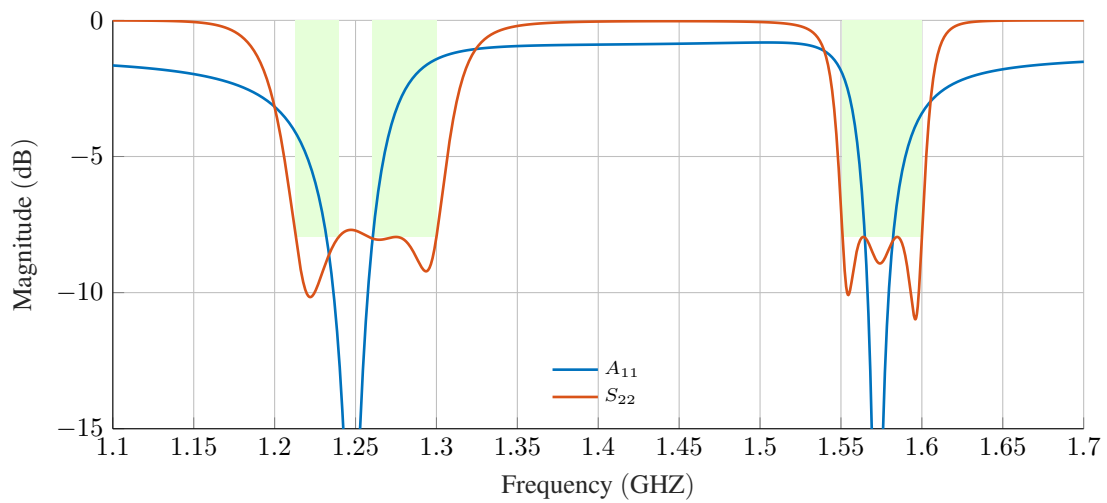


Figure 2: Result of  $\mathcal{P}_C$  with a load of degree 2 and a system of degree 6.

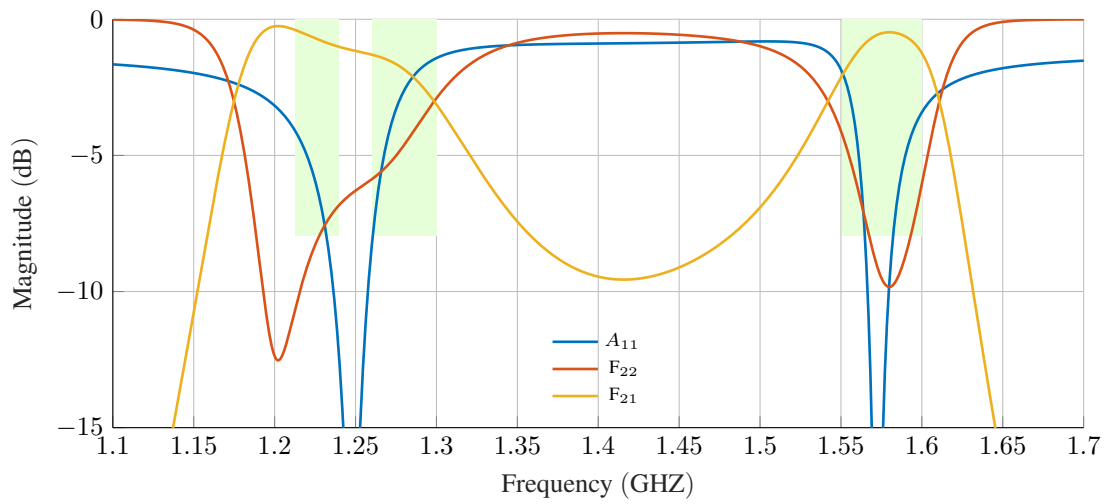


Figure 3: Matching filter providing the response in Fig. 2.

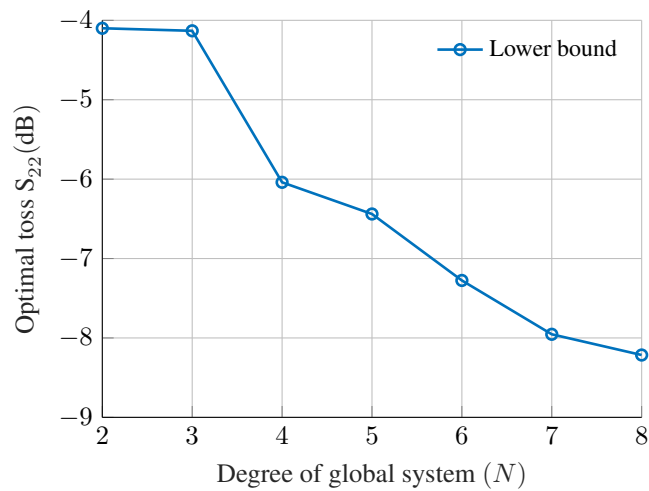


Figure 4: Bounds for the optimal solution of  $\mathcal{P}$ .

### 3.1. EM validation

Finally, we present a practical result considering a microstrip patch antenna for a *GNSS* receiver. The specifications are the coverage of the band  $L_1$  (from 1.55GHz to 1.6GHz). At hand of these specifications, problem  $\mathcal{P}_C$  yields the best possible response  $S_{22}$  of order 4 ( $S_{22}^{OPT}$  in Fig. 6). Note that this is a load of degree 1, therefore the optimal solution to  $\mathcal{P}$  is reached. After de-embedding the antenna at port 2 of the system, the *Belevitch* model of the matching filter of order 3 is obtained. In this example the *Pick matrix* at the optimal point ( $\Delta(P^{opt})$ ) is of rank 0. Therefore the matching filter has no additional transmission zeros and can be implemented by a coupled resonators network with an in-line topology. This filter is realized in *SIW* planar technology fed with *CPWG* input and output lines (Fig. 7) by using the substrate *Rogers RT/duroid 6010LM*. It is important to remark that, in contrast with the traditional filter synthesis, synthesising the right phase of  $F_{22}$  is a crucial point here in order to obtain a filter that is matched at port 2 to the antenna. Therefore, a transmission line of 10.5mm has been required to adjust the phase of  $S_{22}$ .

The practical design of the filter is done via the classical coupling matrix approach [10] using the simulation software *Ansoft Electronic Desktop*. The target matrix  $M_T$  is obtained from the previous algorithm.

$$M_T = \begin{bmatrix} 0 & 1.195 & 0 & 0 & 0 \\ 1.195 & 0 & 1.018 & 0 & 0 \\ 0 & 1.018 & -0.007 & 0.7 & 0 \\ 0 & 0 & 0.7 & -0.404 & 1.009 \\ 0 & 0 & 0 & 1.009 & 0 \end{bmatrix} \quad (14)$$

However this kind of filtering functions differs from the classical *Tchebyshev* responses in the sense that they do not present all reflection zeros distributed on the frequency axis but inside the complex plane. For this reason and in order to achieve a good agreement between the circuital response and the EM response, the design has been assisted with the circuit - extraction software *PRESTO-HF* [14] comparing the target coupling matrix ( $M_T$ ) with the one extracted from the EM response ( $M_{EM}$ ) and adjusting the physical dimensions in consequence. The final error in the coupling matrix is computed as  $E = M_T - M_{EM}$ :

$$E = \begin{bmatrix} 0 & -3.2 & 0 & 0.5 & 0 \\ -3.2 & -0.4 & 0.3 & -0.3 & 0.5 \\ 0 & 0.3 & -1.1 & 0.8 & 0 \\ 0.5 & -0.3 & 0.8 & -0.6 & -2.1 \\ 0 & 0.5 & 0 & -2.1 & 0 \end{bmatrix} \cdot 10^{-2} \quad (15)$$

Fig. 5 shows the comparison between the S-parameters of the filter, obtained in one case from the EM simulation ( $F_{22}^{EM}$  and  $F_{21}^{EM}$ ) and in the other case from the circuital analogue ( $F_{22}^{Goal}$  and  $F_{21}^{Goal}$ ) where a quality factor of  $Q = 200$  has been considered. Moreover, the line  $S_{22}^{EM}$  in Fig. 6 represents the input reflection when the designed *SIW* filter is connected to the antenna.

Note that in this example we implement the filter providing the best matching to the load at port 2. Indeed an excellent match between  $S_{22}^{EM}$  and  $S_{22}^{OPT}$  is obtained in spite of the quality factor of the matching filter  $Q = 200$ . Thus validating the employed synthesis and tuning technique for *SIW* filters. However, due to the finite quality factor, this filter will not provide the optimal transmission loss for the global system. Nevertheless when low quality factors are considered, we can use the optimal response obtained in the lossless case to initialize a local optimisation with the goal of maximising the system efficiency.

## 4. CONCLUSION

A practical implementation of the *Fano-Youla* matching theory by means of convex optimisation has been presented. This approach provides hard lower bounds for the best achievable matching level in a set of frequency bands. Furthermore, if the load to be matched is of degree 1, our algorithm yields the guaranteed best matching response. In this case it is the generalization of the classical quasi-elliptic synthesis technique considering a resistive load to the case of a frequency varying load. Otherwise, for loads of higher degree, our algorithm allows to compute hard lower bounds for the attainable loss level when system  $N$  is considered and provides a rational filter approaching the bound. Finally, a 3-poles *SIW* filter is implemented to match a single-band antenna with an excellent match between the expected result and the *EM* simulation.

## ACKNOWLEDGEMENTS

This work was supported by the French DGA (Direction Générale de l'Armement) and CNES (Centre National d'études Spatiales).



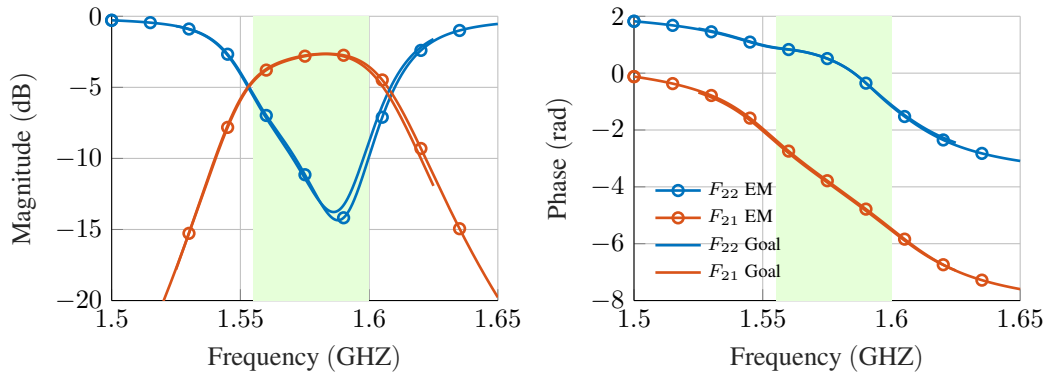


Figure 5: EM simulation of the matching filter and target response.

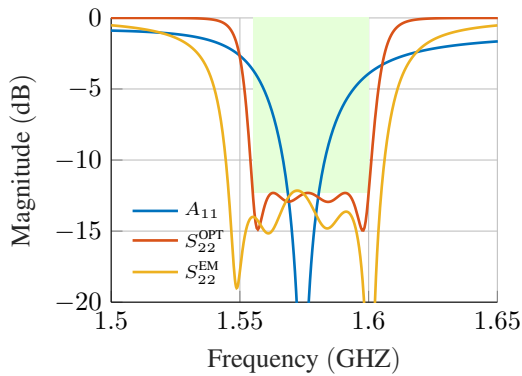


Figure 6: Reflection: synthesis and EM simulation.



Figure 7: Designed SIW filter.

## REFERENCES

- [1] H. Bode, *Network Analysis and Feedback Amplifier Design*. Van Nostrand, 1945.
- [2] R. Fano, "Theoretical limitations on the broadband matching of arbitrary impedances," *Journal of the Franklin Institute*, vol. 249, no. 2, pp. 139 – 154, 1950.
- [3] D. Youla, *A new theory of broadband matching*. Defense Technical Information Center, 1964.
- [4] R. Gudipati and W. K. Chen, "Explicit formulas for the design of broadband matching bandpass equalizers with chebyshev response," in *1995 IEEE International Symposium on Circuits and Systems*, 1995, pp. 1644–1647.
- [5] H. Carlin and P. Civalieri, *Wideband Circuit Design*. Taylor & Francis, 1997.
- [6] J. W. Helton, "Non-euclidean functional analysis and electronics," *Bulletin (New Series) of the American Mathematical Society*, pp. 1–64, 1982.
- [7] S. Darlington, "Synthesis of reactance 4-poles which produce prescribed insertion loss characteristics: Including special applications to filter design," *Journal of Mathematics and Physics*, vol. 18, no. 1-4, pp. 257–353, 1939.
- [8] V. Belevitch, *Classical network theory*. Holden-Day, 1968.
- [9] J. Garnett, *Bounded Analytic Functions*. Academic Press, 1981.
- [10] R. Cameron, R. Mansour, and C. Kudsia, *Microwave Filters for Communication Systems: Fundamentals, Design and Applications*. Wiley, 2007.
- [11] D. Sarason, "Nevanlinna-pick interpolation with boundary data," *Integral Equations and Operator Theory*, vol. 30, pp. 231–250, 06 1998.
- [12] B. Dumitrescu, *Positive Trigonometric Polynomials and Signal Processing Applications*. Springer, Jan. 2007.
- [13] M. Stingl, *On the solution of nonlinear semidefinite programs by augmented Lagrangian methods*. Friedrich Alexander University Nuremberg, 2006.
- [14] Software Presto-HF. Available: <https://project.inria.fr/presto-hf/>.